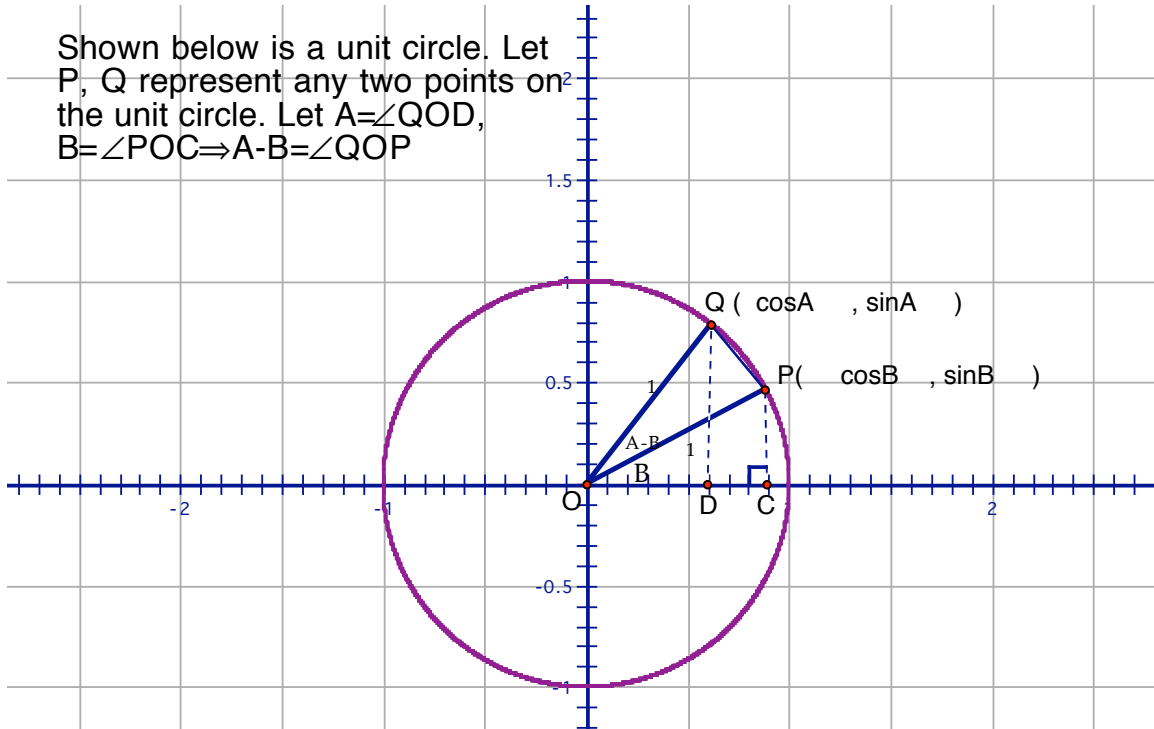


## Sum/Difference, Double Angle Trigonometric Formulae

Using the diagram below, prove the identity:  $\cos(A - B) = \cos A \cos B + \sin A \sin B$

Shown below is a unit circle. Let P, Q represent any two points on the unit circle. Let  $A = \angle QOD$ ,  $B = \angle POC \Rightarrow A - B = \angle QOP$



Proof:  $PQ^2 = 1^2 + 1^2 - 2(1)(1)\cos(A - B)$  from the cosine rule

Also,  $PQ^2 = (\cos A - \cos B)^2 + (\sin A - \sin B)^2$  distance between P, Q.

$\therefore 2 - 2\cos(A - B) = \cos^2 A + \cos^2 B - 2\cos A \cos B + \sin^2 A + \sin^2 B - 2\sin A \sin B$

using the fact that  $\cos^2 A + \sin^2 A = 1$  (for B also) and then dividing by -2, we get

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

Proofs for related sum/difference, double angle formulae:

Listed below are the other related identities. Selected proofs are on next page.

$$1. \cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$1. * \cos(2A) = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$2. \sin(A \pm B) = \sin A \cos B \pm \sin B \cos A$$

$$2 * \sin(2A) = 2\sin A \cos A$$

$$3. \tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$3 * \tan(2A) = \frac{2 \tan A}{1 - \tan^2 A}$$

**Selected Proofs:**

To find  $\cos(A + B)$ : This is also used for sin and tan and is not shown for them.

Think of  $\cos(A + B)$  as  $\cos(A - (-B))$  so replace B with -B in the original identity.

$$\therefore \cos(A + B) = \cos A \cos(-B) - \sin A \sin(-B)$$

$$\text{But, } \cos(-B) = \cos B \text{ and } \sin(-B) = -\sin(B)$$

(Think of the graphs of cos, sin for this)

$$\therefore \cos(A + B) = \cos A \cos B + \sin A \sin B$$

To find  $\sin(A - B)$ , use the cofunction identity.

$$\sin(A - B) = \cos[90 - (A - B)] = \cos[(90 - A) + B]$$

Sub  $90 - A$  and  $B$  into the cosine identity,

$$= \cos(90 - A) \cos B - \sin(90 - A) \sin B$$

$$= \sin A \cos B - \cos A \sin B \text{ as required}$$

$$\therefore \sin(A - B) = \sin A \cos B - \cos A \sin B$$

To find  $\sin 2A$ , just let  $B = A$

To find  $\tan(A + B)$  use  $\frac{\sin(A + B)}{\cos(A + B)}$

$$\frac{\sin(A + B)}{\cos(A + B)} = \frac{\cos B \sin A + \sin B \cos A}{\cos A \cos B - \sin A \sin B}$$

divide each term in the numerator and denominator by  $\cos A \cos B$

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B} \text{ as required. To find } \tan 2A, \text{ replace } B \text{ with } A.$$

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Section B: Long Answer– Full Solutions Required

1. Prove the identity:  $\frac{\tan^2\theta - 1}{\tan^2\theta + 1} = \cos 2\theta$

2.  $\frac{\csc A}{\cot A + \tan A} = \cos A$

3. Solve for  $\theta$  in the interval  $0 \leq \theta \leq 2\pi$ :

a)  $2\cos^2\theta + \cos\theta = 1$

b)  $\sin 2\theta = 3\cos^2\theta$

c)  $2\tan^2\theta + \frac{3}{\cos^2\theta} = 8$

4. a) On the same set of axes graph  $y = \tan x$  and  $y = \sin 2x$  over the interval  $0 \leq x \leq 2\pi$ .

Circle the points of intersection.

b) Find the points of intersections you circled algebraically.

5. Given that  $\frac{\pi}{2} \leq A \leq \pi$  and  $\pi \leq B \leq \frac{3\pi}{2}$  and that  $\sin 2A = -\frac{3}{5}$  and  $\cot B = \frac{5}{12}$ , find:

a)  $\sin A$

b)  $\cos 2B$

c)  $\tan\left(B + \frac{\pi}{4}\right)$ . Use your answer to determine whether B is greater than  $\frac{5\pi}{4}$

**Answers to Section B: Long Answer**

1. 2. are identities 3a)  $\frac{\pi}{3}, \frac{5\pi}{3}, \pi$  b)  $\frac{\pi}{2}, \frac{3\pi}{2}, \tan^{-1}\left(\frac{3}{2}\right), \tan^{-1}\left(\frac{3}{2}\right) + \pi$  c)  $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

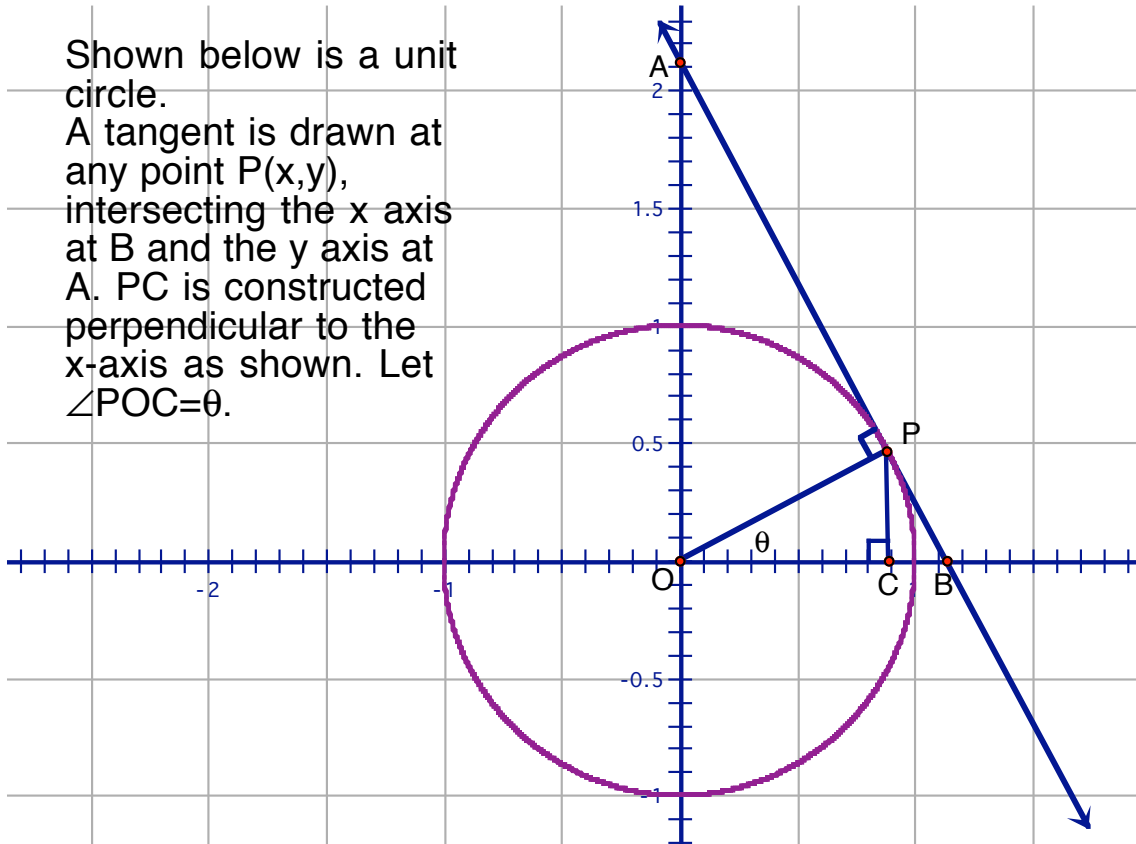
4b)  $x = 0, \pi, 2\pi, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

5.a)  $\frac{1}{\sqrt{10}}$  or  $\frac{3}{\sqrt{10}}$  b)  $-\frac{119}{169}$  c)  $-\frac{17}{7} \because \text{negative} \Rightarrow B > \frac{5\pi}{4}$

10. Find  $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}$  using a geometric proof

Shown below is a unit circle.

A tangent is drawn at any point  $P(x,y)$ , intersecting the  $x$  axis at  $B$  and the  $y$  axis at  $A$ .  $PC$  is constructed perpendicular to the  $x$ -axis as shown. Let  $\angle POC = \theta$ .



The proof is a “sandwich” proof. We will show that the middle part must be between the upper and the lower part, yet at the limiting value will be equal to each of the two.

Consider area  $\triangle OPC \leq \triangle OPB \leq \text{sector OPQ}$  (where  $Q$  is the  $x$ -intercept of the circle)

$$\therefore \text{ in terms of } \theta: \frac{\sin \theta \cos \theta}{2} \leq \frac{1}{2}(1)^2 \theta \leq \frac{\tan \theta}{2}$$

$$\sin \theta \cos \theta \leq \theta \leq \frac{\sin \theta}{\cos \theta}$$

assuming  $\theta$  is acute (so everything is positive)

divide each by  $\sin \theta$

$$\cos \theta \leq \frac{\theta}{\sin \theta} \leq \frac{1}{\cos \theta}$$

In the limiting case when  $\theta \rightarrow 0$ , both  $\cos \theta$  and  $\frac{1}{\cos \theta}$  approach 1.

$$\therefore \lim_{\theta \rightarrow 0} \frac{\theta}{\sin \theta} = 1 \text{ since we have "sandwiched" it between 1 and 1}$$

it follows that  $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$  also, as the reciprocal of 1 equals 1 also.

Problems- Identities, Equations- No calculators! (except for 3c)

- If  $\sin \theta = \frac{2}{3}$  and  $\theta$  is acute, find the value of a)  $\sin 2\theta$  and b)  $\sin 4\theta$ .
- Find  $\tan x$  given that the value of  $\tan\left(x - \frac{3\pi}{4}\right) = 2$
- Solve each of the following in the interval  $[0, 2\pi]$ .
  - $\sin^2 x + \cos^2 x = \cos x$
  - $\sin^2 x \cos^2 x = \frac{3}{16}$
  - $\cos x + \tan x = 0$
  - $\cos 2x = \sin\left(x + \frac{3\pi}{2}\right)$
  - $\tan(2x) = \frac{1}{1 + \tan x}$  (you may use your calculator for this near the end)

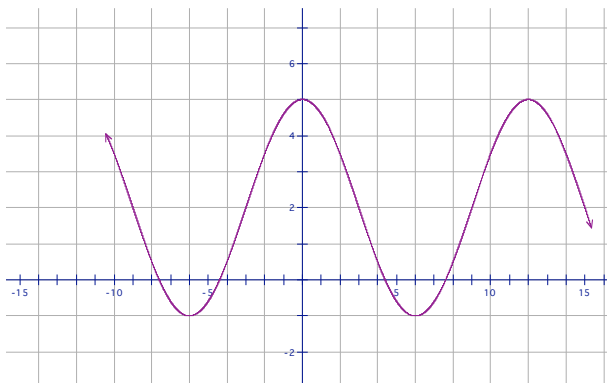
4. Prove each of the following identities:

a)  $\frac{\sin 2x}{1 + \cos 2x} = \tan x$

b)  $2\csc 2x = \sec x \csc x$

5 Predict an equation for the following graph:

Note: the first **minimum** value of  $x > 0$  is at the point  $(6, -1)$

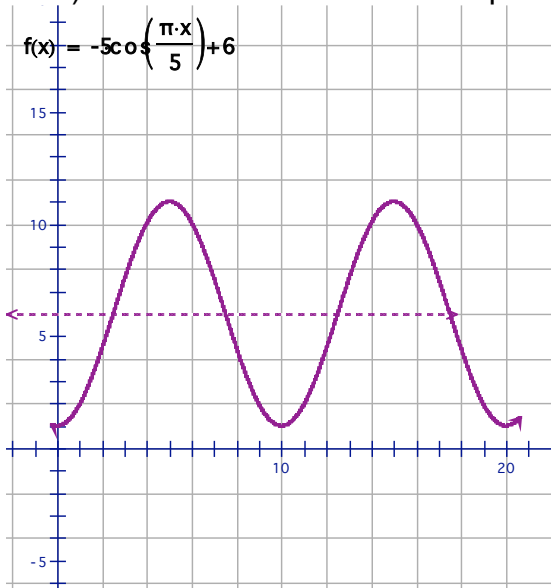


Answers#1-#5:1a)  $\frac{4\sqrt{5}}{9}$  b)  $\frac{8\sqrt{5}}{81}$  2.  $\frac{1}{2}$  3. a)  $0, 2\pi$  b)  $\frac{\pi}{6}, \frac{7\pi}{6}, \frac{\pi}{3}, \frac{4\pi}{3}$  c) 5.617, 3.808

d)  $\frac{\pi}{3}, \frac{5\pi}{3}, \pi$  e)  $.322, \pi + .322$  5.  $y = 3\cos\left(\frac{\pi x}{6}\right) + 2$

6. At the ocean, it is known that the tide follows a trigonometric path. At high tide, the water comes in to a point 1 metre from where I placed a flag. At low tide, the water comes in to a point 11 metres from the same flag. The time it takes from high tide to low tide is 5 hours. It is now midnight and it is high tide.  
 (Note: low tide=max and high tide=min in this case)

a) Plot the motion for two complete cycles below:



b) State a possible equation for this motion.

See graph above.

c) We want to wake up and go to the beach when we can set up our towels at a time between 10 am and 2 pm the next day when the water will be 4 metres from our flag. At what time will this be? Explain what you did, even if you used your graphing calculator to find the answer.

$$\text{let } 4 = -5 \cos\left(\frac{\pi x}{5}\right) + 6$$

$$\frac{2}{5} = \cos\left(\frac{\pi x}{5}\right)$$

$$\Rightarrow \frac{\pi x}{5} = \cos^{-1}\left(\frac{2}{5}\right) + 2k\pi \text{ (we don't need the second case-given } t=10 \text{ to } t=14)$$

$$\therefore x = 11.845 \text{ or } 11:50:42$$

at 11:50:42 am the next day the flag will be 4 m away.