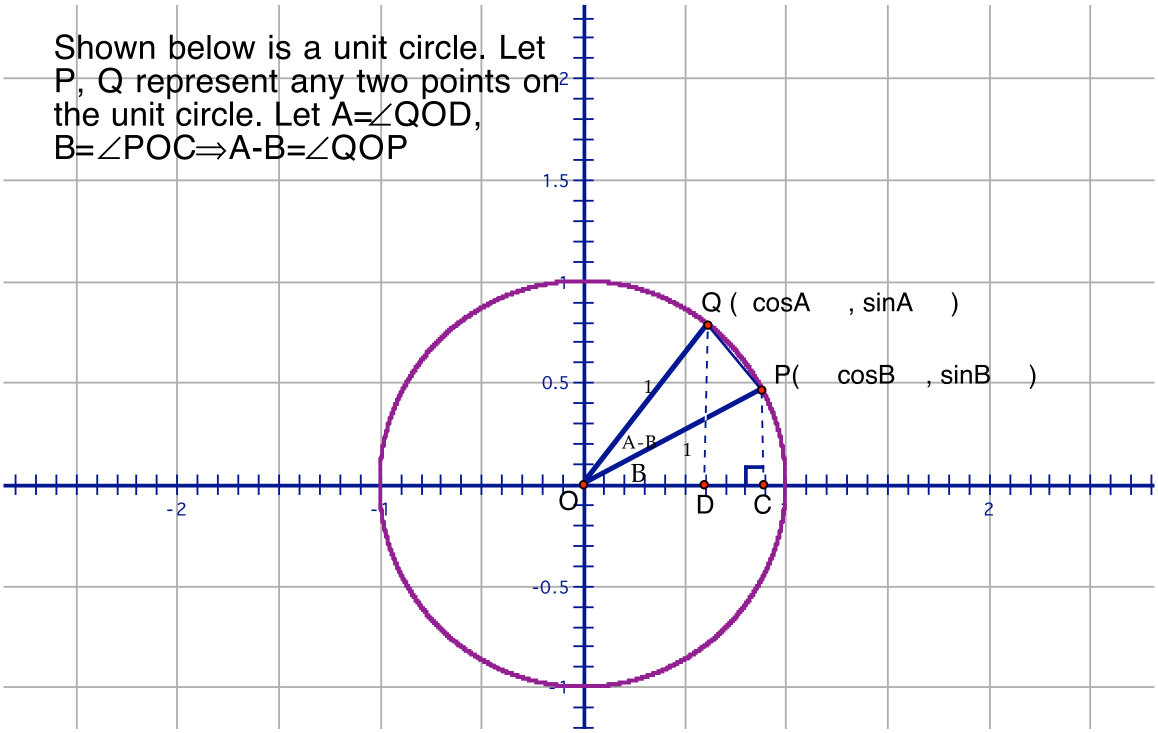
Sum/Difference, Double Angle Trigonometric Formulae

Using the diagram below, prove the identity: $\cos(A-B)=\cos A\cos B+\sin A\sin B$



Proof: $PQ^2=1^2+1^2-2(1)(1)\cos(A-B)$ from the cosine rule

Also, $PQ^2=(\cos A-\cos B)^2+(\sin A-\sin B)^2$ distance between P,Q.

$\therefore 2-2\cos(A-B)=\cos^2 A+\cos^2 B-2\cos A\cos B+\sin^2 A+\sin^2 B-2\sin A\sin B$

using the fact that $\cos^2 A+\sin^2 A=1$ (for B also) and then dividing by -2, we get

$\cos(A-B)=\cos A\cos B+\sin A\sin B$

Proofs for related sum/difference, double angle formulae:

Listed below are the other related identities. Selected proofs are on next page.

$1.\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$

$1.*\cos(2A)=\cos^2 A-\sin^2 A=2\cos^2 A-1=1-2\sin^2 A$

$2.\sin(A\pm B)=\sin A\cos B\pm\sin B\cos A$

$2*\sin(2A)=2\sin A\cos A$

$3.\tan(A\pm B)=\dfrac{\tan A\pm\tan B}{1\mp\tan A\tan B}$

$3*\tan(2A)=\dfrac{2\tan A}{1-\tan^2 A}$

Selected Proofs:

To find $\cos(A+B)$: This is also used for sin and tan and is not shown for them.

Think of $\cos(A+B)$ as $\cos(A-(-B))$ so replace B with -B in the original identity.

$\therefore \cos(A+B) = \cos A \cos(-B) - \sin A \sin(-B)$

But, $\cos(-B) = \cos B$ and $\sin(-B) = -\sin(B)$

(Think of the graphs of cos,sin for this)

$\therefore \cos(A+B) = \cos A \cos B + \sin A \sin B$

To find $\sin(A-B)$, use the cofunction identity.

$\sin(A-B) = \cos[90-(A-B)] = \cos[(90-A)+B]$

Sub $90-A$ and B into the cosine identity,

$= \cos(90-A)\cos B - \sin(90-A)\sin B$

$= \sin A \cos B - \cos A \sin B$ as required

$\therefore \sin(A-B) = \sin A \cos B - \cos A \sin B$

To find $\sin 2A$, *just let* $B = A$

To find $\tan(A+B)$ use $\dfrac{\sin(A+B)}{\cos(A+B)}$

$$\frac{\sin(A+B)}{\cos(A+B)} = \frac{\cos B \sin A + \sin B \cos A}{\cos A \cos B - \sin A \sin B}$$

divide each term in the numerator and denominator by cosAcosB

$= \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ as required. To find tan2A, replace B with A.

Section B: Long Answer – Full Solutions Required

1. Prove the identity: $\frac{\tan^2\theta - 1}{\tan^2\theta + 1} = \cos 2\theta$

2. $\frac{\csc A}{\cot A + \tan A} = \cos A$

3. Solve for θ in the interval $0 \le \theta \le 2\pi$:

a) $2\cos^2\theta + \cos\theta = 1$

b) $\sin 2\theta = 3\cos^2\theta$

c) $2\tan^2\theta + \frac{3}{\cos^2\theta} = 8$

4. a) On the same set of axes graph $y = \tan x$ and $y = \sin 2x$ over the interval $0 \le x \le 2\pi$. Circle the points of intersection.

b) Find the points of intersections you circled algebraically.

5. Given that $\frac{\pi}{2} \le A \le \pi$ and $\pi \le B \le \frac{3\pi}{2}$ and that $\sin 2A = -\frac{3}{5}$ and $\cot B = \frac{5}{12}$, find:

a) $\sin A$

b) $\cos 2B$

c) $\tan\left(B + \frac{\pi}{4}\right)$. Use your answer to determine whether B is greater than $\frac{5\pi}{4}$

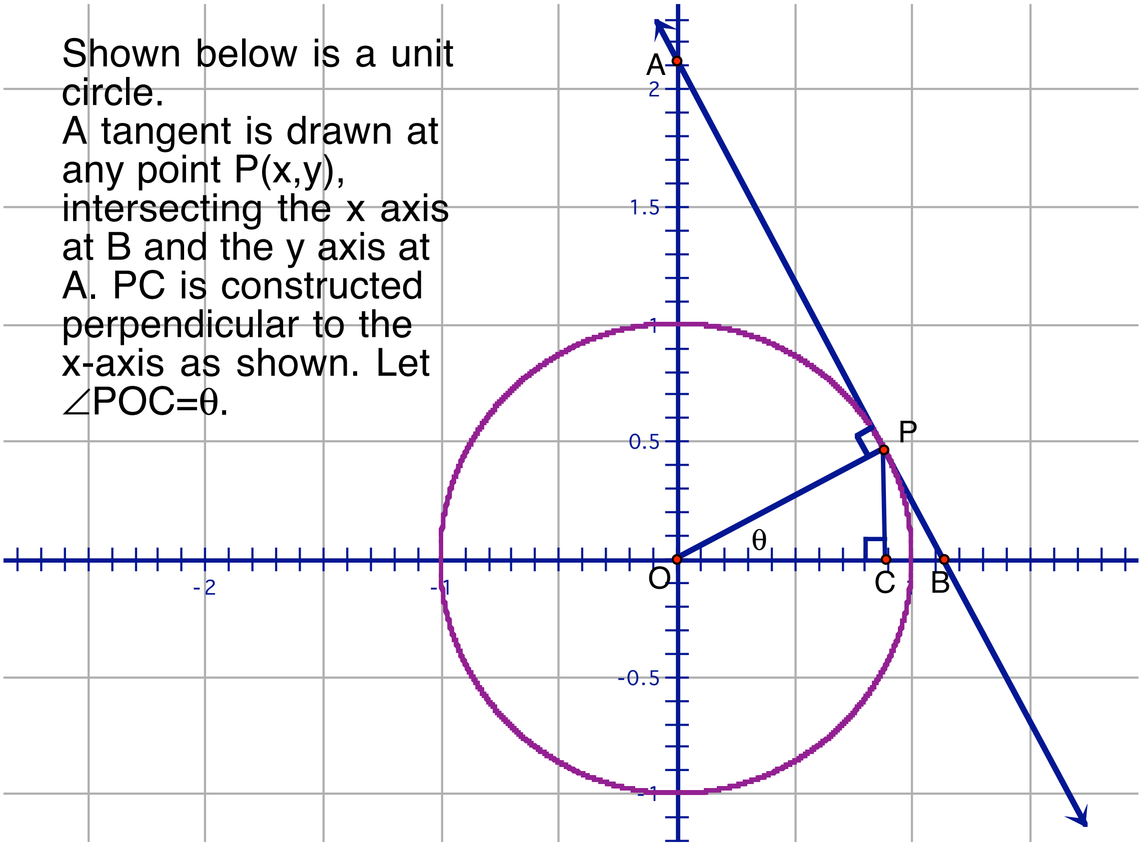
Answers to Section B: Long Answer

1. 2. are identities 3a) $\frac{\pi}{3}, \frac{5\pi}{3}, \pi$ b) $\frac{\pi}{2}, \frac{3\pi}{2}, \tan^{-1}\left(\frac{3}{2}\right), \tan^{-1}\left(\frac{3}{2}\right) + \pi$ c) $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

4b) $x = 0, \pi, 2\pi, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

5.a) $\frac{1}{\sqrt{10}}$ or $\frac{3}{\sqrt{10}}$ b) $-\frac{119}{169}$ c) $-\frac{17}{7} \because$ negative $\Rightarrow B > \frac{5\pi}{4}$

10. Find $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$ using a geometric proof



The proof is a "sandwich" proof. We will show that the middle part must be between the upper and the lower part, yet at the limiting value will be equal to each of the two.

Consider area $\Delta OPC \le \Delta OPB \le$ sectorOPQ (where Q is the xintercept of the circle)

$\therefore$ in terms of θ: $\frac{\sin\theta\cos\theta}{2} \le \frac{1}{2}(1)^2\theta \le \frac{\tan\theta}{2}$

$\sin\theta\cos\theta \le \theta \le \frac{\sin\theta}{\cos\theta}$

assuming θ is acute (so everything is positive)

divide each by $\sin\theta$

$\cos\theta \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}$

In the limiting case when $\theta \to 0$, both $\cos\theta$ and $\frac{1}{\cos\theta}$ approach 1.

$\therefore \lim_{\theta \to 0} \frac{\theta}{\sin\theta} = 1$ since we have "sandwiched" it between 1 and 1

it follows that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ also, as the reciprocal of 1 equals 1 also.

Problems- Identities, Equations- No calculators! (except for 3c)

1. If $\sin\theta = \frac{2}{3}$ and θ is acute, find the value of a) $\sin 2\theta$ and b) $\sin 4\theta$.

2. Find $\tan x$ given that the value of $\tan\left(x - \frac{3\pi}{4}\right) = 2$

3. Solve each of the following in the interval $[0, 2\pi]$.

a) $\sin^2 x + \cos^2 x = \cos x$

b) $\sin^2 x \cos^2 x = \frac{3}{16}$

c) $\cos x + \tan x = 0$

d) $\cos 2x = \sin(x + \frac{3\pi}{2})$

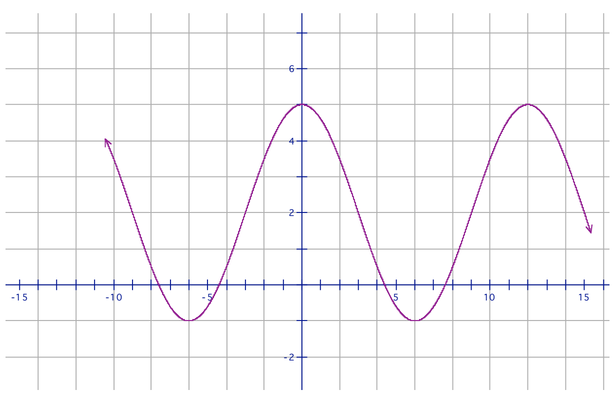
e) $\tan(2x) = \frac{1}{1 + \tan x}$ (you may use your calculator for this near the end)

4. Prove each of the following identities:

a) $\frac{\sin 2x}{1 + \cos 2x} = \tan x$

b) 2csc2x=secxcscx

5 Predict an equation for the following graph:
Note: the first **minimum** value of x>0 is at the point $(6, -1)$

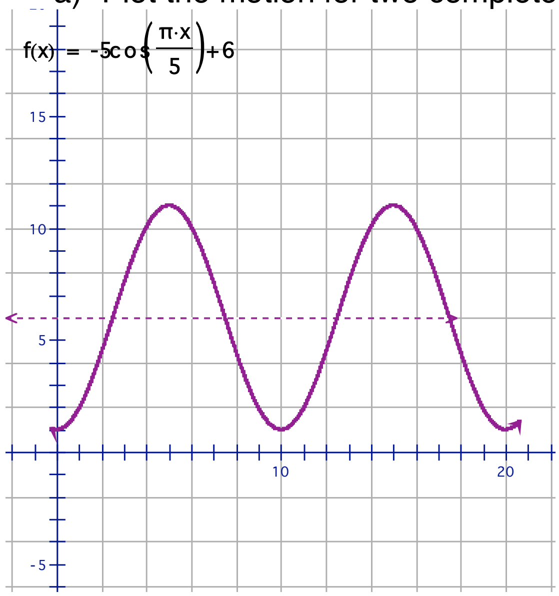


Answers#1-#5:1a) $\frac{4\sqrt{5}}{9}$ b) $\frac{8\sqrt{5}}{81}$ 2. $\frac{1}{2}$ 3. a) $0, 2\pi$ b) $\frac{\pi}{6}, \frac{7\pi}{6}, \frac{\pi}{3}, \frac{4\pi}{3}$ c) 5.617,3.808

d) $\frac{\pi}{3}, \frac{5\pi}{3}, \pi$ e) $.322, \pi + .322$ 5. $y = 3\cos\left(\frac{\pi x}{6}\right) + 2$

6. At the ocean, it is known that the tide follows a trigonometric path. At high tide, the water comes in to a point 1 metre from where I placed a flag. At low tide, the water comes in to a point 11 metres from the same flag. The time it takes from to get from high tide to low tide is 5 hours. It is now midnight and it is high tide. (Note: low tide=max and high tide=min in this case)

a) Plot the motion for two complete cycles below:



b) State a possible equation for this motion.

See graph above.

c) We want to wake up and go to the beach when we can set up our towels at a time between 10 am and 2 pm the next day when the water will be 4 metres from our flag. At what time will this be? Explain what you did, even if you used your graphing calculator to find the answer.

$$\text{let } 4 = -5\cos\left(\frac{\pi x}{5}\right)+6$$

$$\frac{2}{5} = \cos\left(\frac{\pi x}{5}\right)$$

$$\Rightarrow \frac{\pi x}{5} = \cos^{-1}\left(\frac{2}{5}\right)+2k\pi \text{ (we don't need the second case-given t=10to t=14)}$$

$$\therefore x = 11.845 \text{ or } 11{:}50{:}42$$

at 11:50:42 am the next day the flag will be 4 m away.