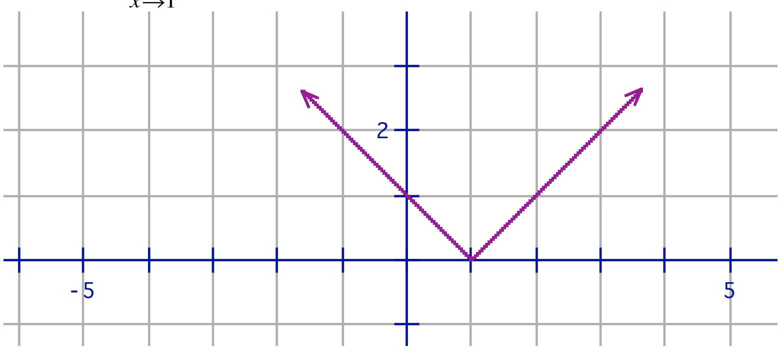
Day 5- Limits, Derivatives by first Principles

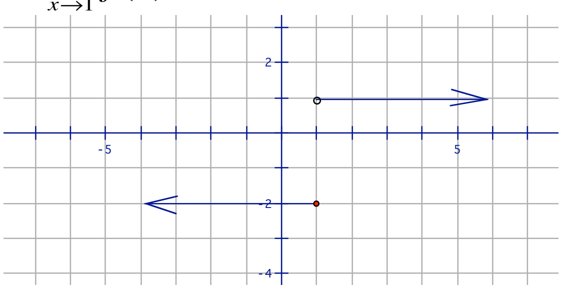
Although we have previously used derivatives in this mini-course, we have done so without regard to how they were found. Let's consider the concept of a limit of a function and how this relates to finding derivatives. This process is called finding the derivative by **first principles**.

The notation for limits is $\lim_{x \to a} f(x)$. Thinking of limits graphically, this simply says that if we look at the graph of $f(x)$, as we get arbitrarily close to the x value of a, if, on both sides of a, the graph approaches the same y value, then this y value is the limit. (So, if y approaches b, then $\lim_{x \to a} f(x) = b$). Note the following graphs below, consider, in each case, $\lim_{x \to 1} f(x)$ as these are the cases we will consider:

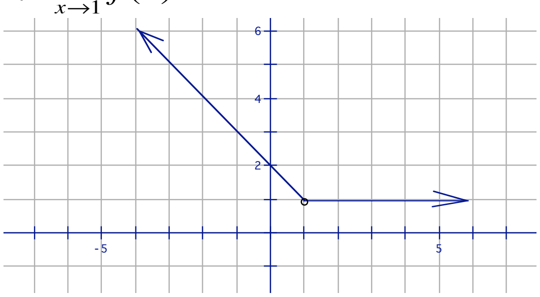
Ex 1 a) $\lim_{x \to 1} f(x) =$ ________ Reason: ____________________



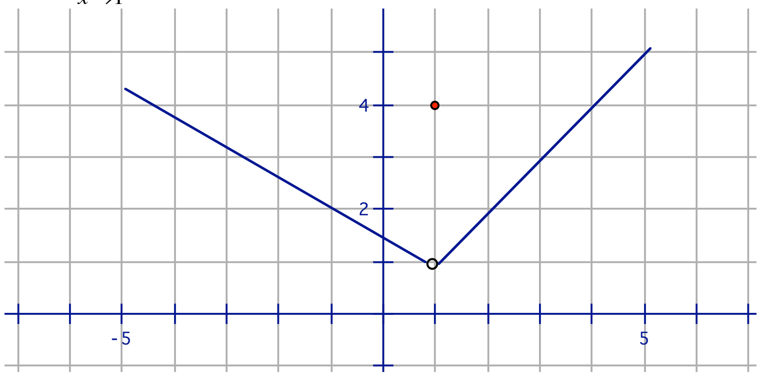
b) $\lim_{x \to 1} f(x) =$ ________ Reason: ____________________



c) $\lim_{x \to 1} f(x) =$ ________ Reason: ____________________



d)) $\lim_{x \to 1} f(x) =$ ________ Reason: ____________________

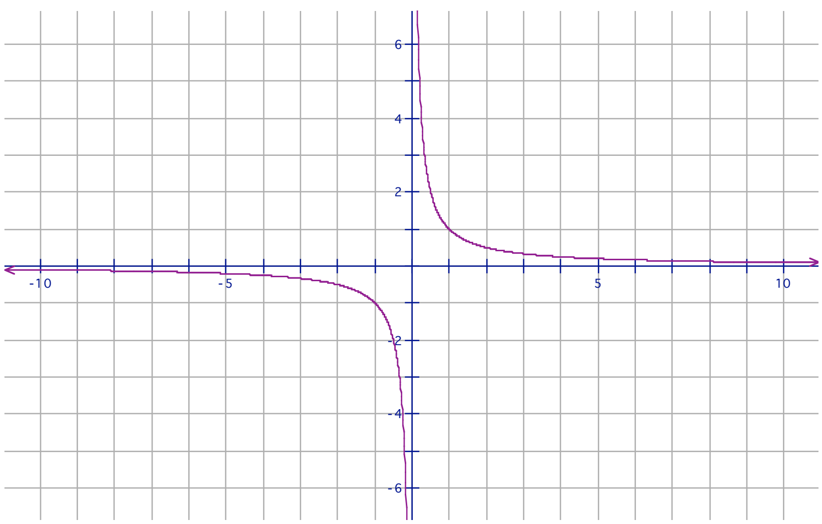


e) $\lim_{x \to 0^+} f(x) =$ ________ Reason: ____________________

$\lim_{x \to 0^-} f(x) =$ ________ Reason: ____________________

$\lim_{x \to \infty} f(x) =$ ________ Reason: ____________________

$\lim_{x \to -\infty} f(x) =$ ________ Reason: ____________________



Algebraically, these can be more difficult to determine:

Ex 3 Evaluate each of the following limits without drawing the graph of the function:

a) $\lim_{x \to 0} \dfrac{x^2 - 9}{x^2 - 4x + 3}$

b) $\lim_{x \to 3} \dfrac{x^2 - 9}{x^2 - 4x + 3}$

c) $\lim\limits_{x \to +\infty} \dfrac{x^2 - 9}{x^2 - 4x + 3}$

d) $\lim\limits_{x \to 1} \dfrac{x^2 - 9}{x^2 - 4x + 3}$

Suppose we try a few more:

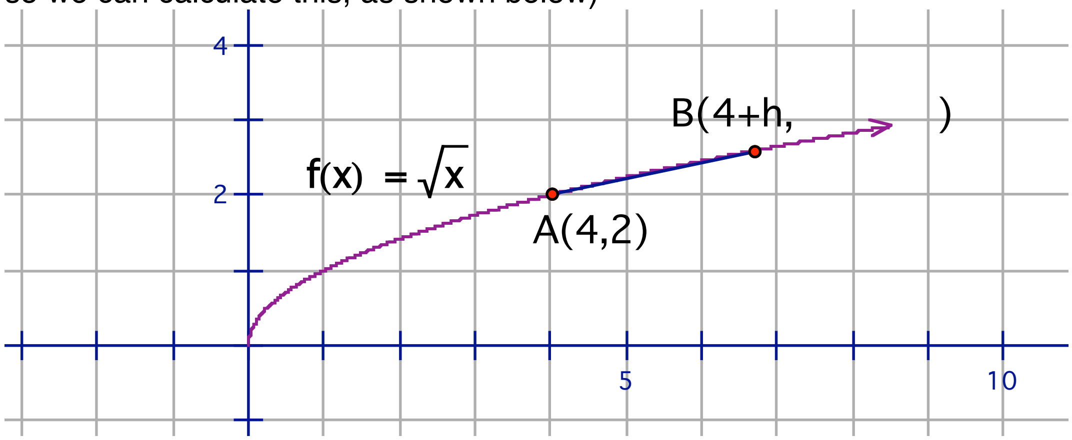
Ex 3

a) $\lim\limits_{x \to 0} \dfrac{\sqrt{x+1} - 2}{x - 3}$

b) $\lim\limits_{x \to 3} \dfrac{\sqrt{x+1} - 2}{x - 3}$

c) $\lim\limits_{x \to -\infty} \dfrac{\sqrt{x+1} - 2}{x - 3}$

Now, limits are used in finding derivatives from first principles as follows. Consider the curve $y=\sqrt{x}$. We wish to find the slope of the tangent (derivative) to this curve at the value of x=4. In order to find the slope of the tangent, we start with finding the slope of a secant (a secant intersects the curve twice, not once, so we can calculate this, as shown below)



We want to find the slope at A, but if we allow $h \to 0$, this cleverly makes B and A virtually the same point, making the secant AB into a tangent at A.
Let's compute mAB .

Example 2: Find the slope of any point at x= a on the curve of $y=\sqrt{x}$

Example 3: Find the derivative of $f(x) = x^3$ a) at $x = 2$ b) at any value of $x = a$

In general, when we want to use limits and first principles to find a derivative, we can write, for any function $f(x)$:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

In general, then, we recall the POWER rule :

If $y = x^n$, where $n \neq 0$, the derivative $\frac{dy}{dx} =$ _____

Continuity and Differentiability

A function $f(x)$ is said to be **continuous** at a point (say x=a) if, when looking at the graph, you don't have to take your pencil off the paper as you pass through that point. In terms of limits, we need two things to be true:

1) $\lim_{x \to a} f(x)$ exists and if this holds, then

2) $\lim_{x \to a} f(x) = f(a)$ (the limit you found equals the value of the function at $x = a$)

You might wonder how it might be that condition 1) is not enough to guarentee continuity ie when $f(a)$ might NOT equal this limit.

Look back at example 1 a)-d) now in light of this.

A function is said to be **differentiable** (and continuous) if both the function and its derivative are continuous. That means that if we graphed the derivative, it would be also be continuous at that point. Go back to Example #1 again. Which of them is/are differentiable at x=1?

Example: If the function defined piecewise as $f(x)=\begin{Bmatrix} x^2+a, \text{ for } x>2 \\ bx^2-3ax, \text{ for } x\le 2 \end{Bmatrix}$ is differentiable at x=2, find the required values of a,b.

Limits using L'Hopital's rule: There is a short cut for testing a limit which is ***indeterminate in form (**gives any of the following results $\frac{0}{0},\infty\cdot 0,\frac{\infty}{\infty}$).

We will discuss a "cheap" proof of this rule, called L'Hopital's Rule. It states that if $\lim_{x\to a}\frac{f(x)}{g(x)}$, upon substituting for x=a gives an indeterminate result of $\frac{0}{0},\infty\cdot 0,\frac{\infty}{\infty}$, then $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$. That is to say, the result will be the same as the limit of the derivatives of the two functions. Of course, this new limit may be "easier" and hence a quick answer may be found, or, it may still be indeterminate and you can then do the same process again. This will eventually give the correct limit, assuming the limit exists (it doesn't always exist). Of course, the limitation to this is that one has to know the derivative of each of the functions first, so it doesn't eliminate the need for first principles.

ex 1 Find $\lim_{x\to 2}\frac{\sqrt{2x}-4x-x^2+10}{3x^2-6x}$

ex 2 Find $\lim_{x\to 1}\dfrac{x-x^2+\ln x}{\sin(\pi x)}$

Proof of L'Hopital's rule for the $\dfrac{0}{0}$ case.

That is, if $f(a)=g(a)=0$, then prove that $\lim_{x\to a}\dfrac{f(x)}{g(x)}=\lim_{x\to a}\dfrac{f'(x)}{g'(x)}$.

Proof:

Problems on Limits, Derivatives by first principles

1. At home, read from AP Calculus*: Chapter 2 p. 26- 35 and more importantly in Chapter 4 p. 119-125 and 138-40 *Book is on Blackboard in Course Documents. Then, try the problems on p. 126, p131 #6, 7, p. 149 #8,9,10

2. Find the derivative a) $f'(1)$ and then b) $f'(x)$ of each of the following functions by first principles using limits.

a) $f(x) = x^3$

b) $f(x) = 2x^2 - x$

c) $f(x) = \dfrac{2}{2x-1}$

d) $f(x) = \dfrac{2}{\sqrt{2x-1}}$

3. Find the following limits a) algebraically and then again by b) using l'Hopital's Rule

a) $\lim\limits_{h\to 0} \dfrac{(9+h)^2 - 3(9+h) - 54}{h}$

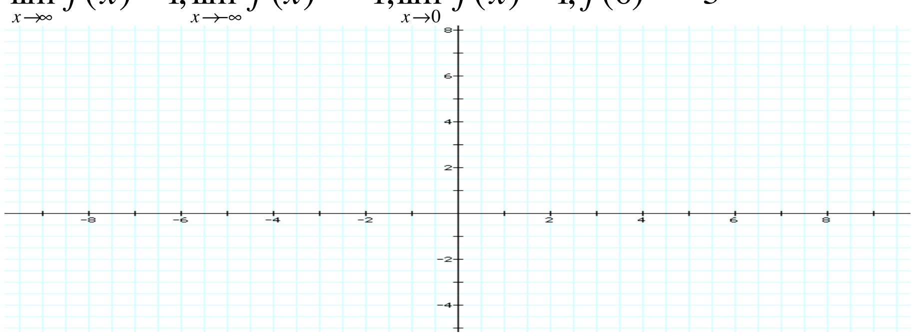
b) $\lim\limits_{x\to 3} \dfrac{\sqrt{x+1} - \sqrt{2x-5}}{x-3}$

c) $\lim\limits_{x\to 2} \dfrac{\frac{1}{x} - \frac{2}{3x-2}}{2-x}$

4. Draw a possible graph below of f(x) containing **each and every one** of the qualities listed below. (there is not a unique solution)

$f(3) = -2, f'(3) = 0, f''(3) > 0, \lim\limits_{x\to 2^+} f(x) = \infty, \lim\limits_{x\to 2^-} f(x) = \infty, \quad f(1) = 3,$

$\lim\limits_{x\to\infty} f(x) = 1, \lim\limits_{x\to -\infty} f(x) = -1, \lim\limits_{x\to 0} f(x) = 1, f(0) = -3$



5.The function f(x) is defined by $\begin{array}{l} f(x) = 2x + b \ \text{ for } \ x < 2 \\ f(x) = \dfrac{a}{x} + x \ \text{for} \quad x \geq 2 \end{array}$.

(a and b are constants). Given that f(x) is continuous and differentiable at x=2, find the values of a and b.

6. If $f(x) = \dfrac{\sqrt{x^2 - 4x - 32}}{x}$, find:

a) $\lim_{x \to \infty} f(x)$

b) $\lim_{x \to -\infty} f(x)$

c) Domain, Range of $f(x)$

7. Let P(a,b) be any point on the parabola $y = x^2$. V(0,0) is the vertex of the parabola. R(0,r) is the point at which the perpendicular bisector of VP intersects the y-axis. Find: $\lim_{a \to 0} r$.

Answers:

1. Answers in text
2. 2a) $3, 3x^2$ b)$3, 4x - 1$ c) $-4, \dfrac{-4}{(2x-1)^2}$ d) $-2, \dfrac{-2}{\sqrt{(2x-1)^3}}$
3. a)15 b) ¼ c) -1/8
4. lots of options
5. a=-4, b=-4
6. a) 1 b) -1
   c) $D = \{x \in \mathbb{R} : x \leq -4 \text{ or } x \geq 8\}$ R=$\{y \in \mathbb{R} : -1.06 \leq y < 1\}$
   note: the graph has a local minimum near $x = -16$
7. ½ (this is challenging)