Derivatives of Exponential, Logarithmic Functions

Given:

if $y = e^x$, then $\frac{dy}{dx} =$ _____ Answer: e^x

if $y = \ln x$ then find $\frac{dy}{dx} =$ ____ Answer: $\frac{1}{x}$

ex1 If $y = xe^x - e^x - 2x^2$, use algebra to determine any relative maximum or minimum values. Is there an absolute minimum or maximum value?

(Answer: local max (0,-1), local min (ln4, 4ln4-2-2(ln4)^2) which is about -2.30)

Ex2 Find the equation of the tangent to

$f(x) = \frac{x}{e}\ln(\frac{e}{x})$ (Answer: $y = \frac{-2}{e}x + 2$)

when

$x = e$

Ex 3 Suppose $h = e^{\ln t} - \ln(t^2 + 1)$. (t in seconds, h in metres).When is the velocity equal to zero? Is this a maximum or minimum height? Find the acceleration. Is there a minimum or maximum velocity?

Answer: a) at t=1 b) neither, as the velocity has a double root at t=1

c) $a = \frac{2(t^2 - 1)}{\left(t^2 + 1\right)^2}$ d) yes, at t=1 the acceleration goes from - to +, so min v at t=1.

Ex 4 Find $\frac{dy}{dx}$ at (1,0) if $\frac{(\ln x)^2}{e^y} - \ln(x+y) = e^{\sin(2x-2)\cos^2 y} - 1$

(Answer: $\frac{dy}{dx} = -3$)

Ex5 Find the derivative of $y = 5^x$. Then try to find a general result.

Answer: a) $\frac{dy}{dx} = (\ln 5)\cdot 5^x$ and in general, $\frac{dy}{dx} = (\ln a)\cdot a^x$, where $y = a^x$

Ex6 Find the derivative of $y = \log_3 x$. Then try to find a general result.

Answer: $\frac{dy}{dx} = \frac{1}{\ln 5}\left(\frac{1}{x}\right)$, and in general, $\frac{dy}{dx} = \frac{1}{\ln a}\left(\frac{1}{x}\right)$, where $y = \log_a x$

Ex 7 How would you find the derivative of $y = x^x$? What is the minimum value of this function?

(Answer: Take ln of both sides first and then take the derivative

min value is $\left(\frac{1}{e}\right)^{\frac{1}{e}} \doteq 0.692$)

Ex 8 Find the smallest positive value of the constant k such that the graphs of $y = e^x$ and $y = k\sin x$ are tangent to one another. Find also the point of tangency. (Answer: $k = \sqrt{2}e^{\frac{\pi}{4}} \doteq 3.10$ and point is $\left(\frac{\pi}{4}, e^{\frac{\pi}{4}}\right)$)

Exponential /Logarithmic Functions- Limits

Exponential limits may still include l'Hopital's rule, order of size, using a substitution and first principles of derivatives as methods of solution, but there are two new types of limits we will discuss involving **definition of e^x and logarithmic method.**

Ex 1 Evaluate the following limits:

a) $\lim\limits_{x \to \infty} \dfrac{2^x + x}{2^{x+1} + x^2}$ b) $\lim\limits_{x \to \pi} \dfrac{\ln(\sin\frac{x}{2})}{\cos x + 1}$ c) $\lim\limits_{x \to \infty} x\ln(\frac{1}{x} + 1)$

d) $\lim\limits_{x \to \infty} (1 + \frac{1}{2x})^x$ e) $\lim\limits_{x \to \infty} x(1 - \frac{1}{x})^x$ f) $\lim\limits_{x \to \infty} 4xe^{1/x} - 4x$

Answers: a) 0 b) $-\frac{1}{4}$ c) 1 d) $\sqrt{e}$ e) no limit f) 4

Day 9- Problems- Logarithms, Exponential Functions

1. Solve for x without a calculator. Leave answers in exact form:
   a) $6+e^{x}=16e^{-x}$ b) $\ln x+\log_{e^2} x=-3$

2. Find the equation of the tangent to $y=3^{2x-2}-3^{x-1}-(\ln 3)x+\ln 3+2x$ at x=1. Leave answers in exact form.

3. Find the value of the constant k such that the graphs of $y=\ln x$ and $y=-\frac{1}{3}x^{3}+k$ will intersect at right angles.

4. The graphs of $f(x)=x^{2}$ and $g(x)=k\ln x$, where k is a positive constant are tangent to one another. Find the point of tangency and the exact value of k.

5. If $e^{xy}=2$, then at the point (1,ln2), $\frac{dy}{dx}=$

a) –ln2 b) 2ln2 c) ln2 d) –2e e) –4ln2

6. If $f(x)=e^{-x}+2$ for $x<0$ and $f(x)=ax+b$ for $x\geq 0$, and if f(x) is differentiable at x=0, then a+b=

a) 0 b) 1 c) 2 d) 3 e) 4

7. Evaluate: a) $\lim_{x\to-\infty} e^{x}\ln(1-x)$ b) $\lim_{x\to 0}(2\cos x-1)^{\frac{1}{x}}$

8.
Find any value of k, where k is a constant, such that the graphs of $y=k^{x}$ and $y=\log_{k} x$ will be tangent to one another. Find the point of tangency.

(Solution follows on next page)

Answers: 1. a) ln 2 b) $\frac{1}{e^2}$ 2. $y = 2x$ 3. $k = \frac{1}{3}$ 4. point $(\sqrt{e}, e)$ and $k = 2e$

5. A 6. C (a= -1, b=3) 7. a) 0 b) 1

Full Solution to #8

$y = \log_k x$ eq1 $\Leftrightarrow y = \frac{\ln x}{\ln k}$

$y = k^x$ eq2

let $k^x = \frac{\ln x}{\ln k}$ eq3

also, slopes are equal,

$\therefore \ln k(k^x) = \frac{1}{x \ln k} \Leftrightarrow k^x = \frac{1}{x(\ln k)^2}$ eq4

Equating,

$$\frac{1}{x(\ln k)^2} = \frac{\ln x}{\ln k}$$

$$\ln k = \frac{1}{x \ln x} \Rightarrow k = e^{1/(x \ln x)} = e^{(x \ln x)^{-1}}$$

But in order to maximise k, let $\frac{dk}{dx} = 0$

$$0 = e^{1/x \ln x}(-1)(x \ln x)^{-2}[\ln x + 1]$$

$$\Rightarrow \ln x + 1 = 0 \Rightarrow x = \frac{1}{e}$$

$\therefore$ the point of tangency is $\left(\frac{1}{e}, e^{1/(\ln x)}\right)$