CHAPTER 6

Maximum / Minimum Problems

Methods for solving practical maximum or minimum problems will be examined by examples.

Example

Question: The material for the square base of a rectangular box with open top costs 27¢ per square cm. and for the other faces costs $1\frac{1}{2}$¢ per square cm. Find the dimensions of such a box of maximum volume which can be made for $65.61.

Answer:

Let the dimensions of the box be $x$ cms by $x$ cms by $h$ cms as shown.

Cost of making the box is $27x^2 + \left(13\frac{1}{2}\right)4xh$

\[\text{(base)} \quad \text{(4 faces)}\]

\[\therefore \quad 27x^2 + 54xh = 6561 \quad (1)\]

i.e. $x^2 + 2xh = 243 \quad (1)$
We wish to maximize the volume \( (V) \) and

\[
V = x^2h
\]  
\hspace{1cm} (2)

Substituting from (1) into (2) yields:

\[
V = x^2 \left( \frac{243 - x^2}{2x} \right) = \frac{1}{2} (243x - x^3) \]  
\hspace{1cm} (2)

For our purposes, clearly volume and width are positive and a graph of volume against length of the base would look like:

The maximum value occurs when \( \frac{dV}{dx} = 0 \)

i.e. differentiating (2) with respect to \( x \) yields

\[
\frac{dV}{dx} = \frac{1}{2} (243 - 3x^2)
\]

When \( \frac{dV}{dx} = 0 \), \( x = 9 \) which clearly yields a maximum volume as seen on the graph. By substituting for \( x \) in (1) it follows that \( h = 9 \) also.

Therefore the box of maximum volume is 9 cms by 9 cms by 9 cms.
**Example**

**Question:** Find the point(s) on the graph of \( y = x^2 \) which is (are) nearest to \( A \left( 0, \frac{1}{2} \right) \).

Let \( P \) be a point \((x, y)\) on the graph of \( y = x^2 \). Let \( AP = s \).

Then \( s = \sqrt{(x-0)^2 + \left( y - \frac{1}{2} \right)^2} \)

But \( P \) lies on \( y = x^2 \).

\[
\therefore \quad s = \sqrt{x^2 + \left( x^2 - \frac{1}{2} \right)^2} \quad (1)
\]

Note that it would easier to write (1) as \( s^2 = x^2 + \left( x^2 - \frac{1}{2} \right)^2 \) (1) for differentiating purposes.
We wish to minimize $s$ and hence we need to differentiate (1) with respect to $x$.

i.e. \[ 2s \frac{ds}{dx} = 2x + 2 \left( x^2 - \frac{1}{2} \right) 2x \quad (1)' \]

When $s$ is a relative minimum, $\frac{ds}{dx} = 0$.

i.e. \[ 0 = 2x + (2x^2 - 3)2x \]
\[ = 2x(1 + 2x^2 - 3) \]
\[ = 2x(2x^2 - 2) \]
\[ = 4x(x-1)(x+1) \]
i.e. \[ x = 0, \; x = 1, \; \text{or} \; x = -1. \]
i.e. \[ P \text{ is } (0,0), \; (1,1), \; \text{or} \; (-1,1). \]

Comparing the three distances from $A$ to the three possible positions of $P$, it is clear that the minimum distance occurs when $P$ is either $(1,1)$ or $(-1,1)$.

Note that, in fact, the distance from $(0,0)$ to $A$ is a relative maximum.
Sometimes a maximum/minimum question is best answered differentiating more
than one equation.

**Example**

*Question:* Prove that the rectangle of largest area which can be inscribed in a
circle of fixed radius is a square.

*Answer:*

Let the fixed radius of the circle be *r* and the variable dimensions of
the rectangle be *l* (length) and *w* (width).

Let *A* be the area of the rectangle.

Then \[ A = lw \] \hspace{1cm} (1)

And \[ 4r^2 = l^2 + w^2 \] \hspace{1cm} (2) Pythagoras.
Differentiate both equations with respect to \( w \).

\[
\frac{dA}{dw} = w \frac{dl}{dw} + l \quad (1)'
\]

\[
0 = 2l \frac{dl}{dw} + 2w \quad (2)'
\]

From (2)', \( \frac{dl}{dw} = -\frac{w}{l} \)

Substituting in (1)' yields

\[
\frac{dA}{dw} = -\frac{w^2}{l} + l \quad (1)'
\]

When \( A \) is maximum, \( \frac{dA}{dw} = 0 \)

i.e. \( 0 = -\frac{w^2}{l} + l \)

and hence \( w = l \).

\[ \therefore \text{ The rectangle of maximum area is a square.} \]
Example

Question: A wire of length 60 metres is cut into two pieces. One piece is bent into the shape of an equilateral triangle and the other piece is bent into a square. What are the lengths of each side of the triangle and square so the total area of the triangle and the square is minimized (and maximized?)

Answer:

Let each side of the triangle and square be $a$ metres and $b$ metres respectively as shown.

Let $A$ be the total area and

then \[ A = \frac{\sqrt{3}}{4} a^2 + b^2 \] (1)

and \[ 60 = 3a + 4b \] (2)

Differentiate each equation with respect to $a$.

\[ \frac{dA}{da} = \frac{\sqrt{3}}{2} a + 2b \frac{db}{da} \] (1)'

\[ 0 = 3 + 4 \frac{db}{da} \] (2)'

Substituting for \(\frac{db}{da}\) from (2)' into (1)' yields

\[
\frac{dA}{da} = \frac{\sqrt{3}}{2} a + 2b \left( \frac{-3}{4} \right) \quad (1)'
\]

\[
= \frac{\sqrt{3}}{2} \left( a - \sqrt{3}b \right) \quad (1)'
\]

Since \(A\) is to be minimized (or maximized)

Let \(\frac{dA}{da} = 0\) \quad i.e. \quad \(a = \sqrt{3}b\)

and hence substituting in (2) yields \(a = 11.3\) (approx.) and \(b = 6.524\).

i.e. Length of side of the triangle is 11.3 and length of the side of the square is 6.524.

It is not however readily clear whether these dimensions produce a maximum or minimum total area or possibly only a critical value.

It is clear that a finite length of wire can be a boundary for only a finite area and hence a maximum area must exist as indeed a minimum area must exist also.

Substituting for \(b\) from (2) into (1) yields

\[
A = \frac{\sqrt{3}}{4} a^2 + \left( \frac{60 - 3a}{4} \right)^2
\]

\[
A \approx 0.9955a^2 - 22.5a + 225
\]
Note that \( 0 \leq a \leq 20 \) (a bounded domain) and hence from the graph it is clear that the maximum or minimum total area occurs when \( \frac{dA}{da} \) is not zero, it occurs at an end point of the graph.

To investigate we need to look at the graph of area against the side of the triangle. The graph clearly illustrates that the minimum total area occurs when \( a = 11.3 \) (approx.) as found earlier and the maximum total area occurs when \( a = 0 \) i.e. when the piece of wire is bent entirely into a square (15 by 15) to yield a maximum area of 225 square cms.
Worksheet 1

Max/Min Problems

1. Find the maximum volume of a cylinder whose radius and height add up to 24.

2. The sum of two numbers is 4. Find the maximum value of $xy^3$ where $x$ and $y$ are the numbers.

3. A rectangular box is to have a capacity of 72 cubic centimetres. If the box is twice as long as it is wide, find the dimensions of the box which require the least material.

4. The volume of a cone is $18\pi$ cubic metres. Find the minimum length of the slant edge.

5. The slant edge of a cone is $3\sqrt{3}$. Find the height of the cone when the volume is a maximum.

6. Find the minimum value of $x - \frac{8}{3}\sqrt{3x+1}$. Does it have a maximum value?

7. The material for the bottom of a rectangular box with square base and open top costs 3¢ per sq. cm. and for the other faces costs 2¢ per sq. cm. Find the dimensions of such a box of maximum volume which can be made for $5.76.

8. If $xy = 48$ find the minimum value of $x + y^3$ for positive of $x$ and $y$.

9. Find the dimensions of the cylinder of maximum volume which can be inscribed in a sphere of radius 3 cms.
10. A rectangular sheet of cardboard is 8 cms by 5 cms. Equal squares are cut from each of the corners so that the remainder can be folded into an open topped box. Find the maximum volume of the box.

11. Find the maximum volume of a cylinder which can be inscribed in a cone whose height is 3 cms and whose base radius is 3 cms.

12. The volume of a closed rectangular box with square base is 27 cubic metres. Find the minimum total surface area of the box. Is there a maximum surface area?

13. Find the dimensions of the cone of maximum volume which can be inscribed in a sphere of radius 12 cms.

14. A closed metal box has a square base and top. The square base and top cost $2 per square metre, but the other faces cost $4 per square metre. The minimum cost of such a box having a volume of 4 cubic metres is:

(A) $2   (B) $8   (C) $16   (D) $48   (E) $64

Answers to Worksheet 1

1. $2048\pi$  
2. 27  
3. 3 by 4 by 6  
4. $3\sqrt{3}$  
5. 3  
6. $-\frac{17}{3}$  No  
7. 8 by 8 by 6  
8. 32  
9. $r = \sqrt{6}, \quad h = 2\sqrt{3}$  height is 16  
10. 18 cubic cms  
11. $4\pi$  
12. 54  No  
13. radius of base is $8\sqrt{2}$  
14. D
Worksheet 2

**Max/Min Problems**

1. At 12 noon a ship going due east at 12 knots crosses 10 nautical miles ahead of a second ship going due north at 16 knots.
   a) If $s$ is the number of nautical miles separating the ships, express $s$ in terms of $t$ (the number of hours after 12 noon).
   b) When are the ships closest and what is the least distance between them?

2. Find the minimum distance of a point on the graph $xy^2 = 16$ from the origin.

3. A man can row at 3 m.p.h. and run at 5 m.p.h. He is 5 miles out to sea and wishes to get to a point on the coast 13 miles from where he is now. Where should he land on the coast to get there as soon as possible? Does it matter how far the point on the coast is from the man?

4. Find the dimensions of the rectangle of maximum area in the first quadrant with vertices on the $x$ axis, on the $y$ axis, at the origin and on the parabola $y = 36 - x^2$.

5. Find the maximum volume of a cylinder which can be placed inside a frustrum (lampshade) whose height is 4 cms and whose radii are 1 cms and 3 cms. Is there a minimum volume for the cylinder? If so, what is the radius of that cylinder?

6. A rectangle is to have an area of 32 square cms. Find its dimensions so that the distance from one corner to the mid point of a non-adjacent edge is a minimum.
7. A poster is to contain 50 square cms. of printed matter with margins of 4 cms.
each at the top and bottom and 2 cms at each side. Find the overall
dimensions if the total area is a minimum. Does the poster have a maximum
area?

8. A cylinder has a total external surface area of $54\pi$ square cms. Find the
maximum volume of the cylinder.

9. Find the shortest distance between $y = x + 10$ and $y = 6\sqrt{x}$.

Answers to Worksheet 2

1. a) $s^2 = (12t)^2 + (10 - 16t)^2$ 
   b) 12:24 p.m. 6 miles

2. $2\sqrt{3}$

3. $\frac{3}{4}$ miles No

4. $2\sqrt{3}$ by 24

5. $8\pi$. Yes, $r = 3$.

6. 4 by 8

7. 18 by 9 No

8. $54\pi$ cubic cms

9. $\frac{1}{\sqrt{2}}$
Worksheet 3

**Max/Min Problems**

1. $ABCD$ is a trapezoid in which $AB$ is parallel to $DC$. $AB = BC = AD = 10$ cms. Find $CD$ so that the area of the trapezoid is maximized and find the maximum area.

2. A rectangle has constant area. Show that the length of a diagonal is least when the rectangle is a square.

3. A sailing ship is 25 nautical miles due north of a floating barge. If the sailing ship sails south at 4 knots while the barge floats east at 3 knots find the minimum distance between them.

4. A sector of a circle has fixed perimeter. For what central angle $\theta$ (in radians) will the area be greatest?

5. The cost of laying cable on land is $2 per metre and the cost of laying cable under water is $3 per metre. In the diagram below the river is 50 metres wide and the distance $AC$ is 100 metres. Find the location of $P$ if the cost of laying the cable from $A$ to $B$ is a minimum.
6. A right circular cylindrical can is to have a volume of $90\pi$ cubic cms. Find the height $h$ and the radius $r$ such that the cost of the can will be a minimum given that the top and bottom cost 5¢ per square cm. and the lateral surface area costs 3¢ per square cm.

7. Find the maximum area of a rectangle $MNPQ$ where P and Q are two points on the graph of $y=\frac{8}{1+x^2}$ and N and M are the two corresponding points on the $x$ axis.

8. Using a graphing calculator, find the approximate position of the point(s) on the curve $y=x^2-4x+10$ between (0,10) and (4,10)
   a) nearest to (1,6).
   b) farthest from (1,6).

Answers to Worksheet 3

1. 20 cms, $75\sqrt{3}$
2. ---
3. 15 miles
4. 2 radians
5. 44.7 miles from C
6. $h=10$ and $r=3$
7. 8
8. a) (1.41, 6.35) b) (4, 10)
Worksheet 4

**Max/Min Problems**

1. A piece of wire 8 metres long is cut into two pieces. One piece is bent into the shape of a circle and the other into the shape of a square. Find the radius of the circle so that the sum of the two areas is a minimum. Is there a maximum area?

2. Find the dimensions of the rectangle of maximum area in the first quadrant with vertices on the $x$ axis, on the $y$ axis, at the origin and on the parabola $y = 75 - x^2$.

3. A cone has altitude 12 cms. and a base radius of 6 cms. Another cone is inscribed inside the first cone with its vertex at the center of the base of the first cone and its base parallel to the base of the first cone. Find the dimensions of maximum volume.

4. Find the proportions of a right circular cylinder of greatest volume which can be inscribed inside a sphere of radius $r$.

5. The cost of fuel (per hour) in running a locomotive is proportional to the square of the speed and is $25$ per hour for a speed of $25$ m.p.h. Other costs amount to $100$ per hour regardless of the speed. Find the speed at which the motorist will make the cost per mile a minimum.
6. A motorist is stranded in a desert 5 kms. from a point A, which is the point on a long straight road nearest to him. He wishes to get to a point B, on the road, which is 5 kms. from A. If he can travel at 15 km per hour on the desert and 39 km per hour on the road, find the point at which he must hit the road to get to B in the shortest possible time.

7. A man is 3 miles out to sea from the nearest point A on land on a straight coastline. He can row at 4 m.p.h. and he can jog at k m.p.h. What is his jogging speed if he wishes to reach some point B on the coast as quickly as possible and he therefore lands 4 miles from A? Assume that B is at least 4 miles from A. Note that the distance AB is not relevant.

8. a) Find the point Q on the curve defined by \( x^2 - y^2 = 16 \) in the interval \( 4 \leq x \leq 5 \) which is nearest to point P \((0,2)\).

b) Find the point on the curve in the same interval that is most distant from P.

c) Verify that PQ is perpendicular to the tangent to the curve at point Q.

Answers to Worksheet 4

1. \( \frac{4}{4+\pi} \)

2. 5 by 50

3. \( r = 4, h = 4 \)

4. \( h:r = \sqrt{2}:1 \)

5. 50 m.p.h

6. \( \frac{25}{12} \)

7. 5 m.p.h.

8. a) \( (\sqrt{17},1) \)
Worksheet 5

**Max/Min Problems**

1. An isosceles triangle is circumscribed about a circle of radius 3 cms. Find the minimum possible area of the triangle.

2. Find the point on the graph of \( y = \sqrt{x} \) which is nearest to \((1,0)\).

3. A variable line through the point \((1,2)\) intersects the \(x\) axis at \((a,0)\) and intersects the \(y\) axis at \((0,b)\). These points are A and B respectively. Find the minimum area of triangle \(AOB\) if O is the origin and \(a\) and \(b\) are positive.

4. 34 feet of wire are to be divided up into two separate pieces, one of which is made into a square and the other into a rectangle which is twice as long as it is wide. Find the minimum total area and the maximum total area.

5. What are the dimensions of a rectangle of greatest area which can be laid out in an isosceles triangle with base 36 cms. and height 12 cms.

6. The cost of fuel required to operate a boat at a speed of \(r\) m.p.h. through the water is \(0.05r^2\) dollars per hour. If the operator charges $3 per hour for the use of the boat, what is the most economically way, in dollars per mile, to travel upstream against a current of 2 m.p.h.

7. A cylindrical vessel with circular base is closed at both ends. If its volume is 10 cubic centimetres find the base radius when the total external surface is least.

8. Find the point(s) on \(x^2 - y^2 = 4\) which are closest to \((6,0)\).
Answers to Worksheet 5

1. $27\sqrt{3}$ square cms

2. $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$

3. 4

4. 34 square ft and $72\frac{1}{4}$ square ft

5. 18 by 6

6. 10 m.p.h. through the water

7. $r = 1.17$ (approx.)

8. $(3, \sqrt{5})$ and $(3, -\sqrt{5})$
Worksheet 6

1. A man rows 3 miles out to sea from point A on a straight coast. He then wishes to get as quickly as possible to point B on the coast 10 miles from A. He can row at 4 m.p.h. and run at 5 m.p.h. How far from A should he land?

2. An open-topped storage box is to have a square base and vertical faces. If the amount of sheet metal available is fixed, find the most efficient shape to maximize the volume.

3. A teepee is to be made of poles which are 6 metres long. What radius will achieve the teepee of maximum volume?

4. Find the dimensions of the right circular cone of minimum volume which can be circumscribed about a sphere of radius 8 cms.

5. A rectangular sheet of cardboard of length 4 metres and width 2.5 metres has four equal squares cut away from its corners and the resulting sheet is folded to form an open-topped box. Find the maximum volume of the box.

Answers to Worksheet 6

1. 4 miles

2. $h$ by $2h$ by $2h$ where $h$

3. $2\sqrt{6}$

4. $h = 32$ and $r = 8\sqrt{2}$

5. $2\frac{1}{4}$ cubic m.