Chapter Four

Linear Dependence and Independence of Vectors

Note: - Throughout this chapter all vectors, **a,b,c** are assumed non-zero vectors.

4.1 Definition Linear dependence

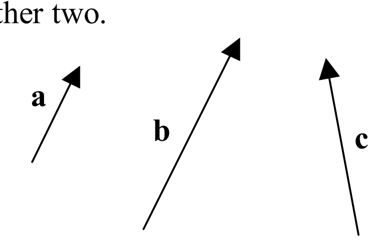
Two vectors are said to be **LINEARLY DEPENDENT** if one is a scalar multiple of the other, i.e. **a** = m**b**. This means that **a** and **b** are parallel vectors (or collinear – but remember position of vectors is not significant).

Definition

Three vectors **a**, **b** and **c** are **LINEARLY DEPENDENT** if it is possible to write one of the vectors in terms of the other two, e.g. **a** = m**b** + n**c**

Note: - This does not mean necessarily that it is possible to write each one of **a,b,c** in terms of the other two.

For example for



$\mathbf{b} = \frac{3}{2}\mathbf{a}$ Here $\mathbf{b} = \frac{3}{2}\mathbf{a} + 0\mathbf{c}$ hence

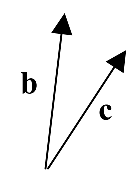
{**a,b,c**} is a linearly dependent set of vectors but it is not possible to write **c** in terms of **a** and **b**. If we say **a** = m**b** + n**c** then **a** is a linear combination of **b** and **c**.

From here on, we shall abbreviate linearly dependent to dependent.

Theorem 4.1 'Any **three dependent** vectors are **coplanar**'

Proof

Let **b** and **c** be two arbitrary vectors which can be assumed without loss of generality to lie in the plane of this page (remember position of vectors is not significant and hence they can be translated to have a common 'tail point' and hence lie in a common plane).



Then clearly $m\mathbf{b}$ and $n\mathbf{c}$ individually lie in the plane of this page and by definition of addition of vectors it follows that $\mathbf{a} = (m\mathbf{b} + n\mathbf{c})$ also lies in the plane of this page.

Any linear combination of two vectors lies in the plane of those two vectors.

Exercise 4.1

1. Let **a** and **b** be two non-parallel vectors.
   i) Is $\{\mathbf{a} + 2\mathbf{b}, \mathbf{a} + \mathbf{b}\}$ a dependent set of vectors?
   ii) Is $\{\mathbf{a} + \mathbf{b}, 2\mathbf{a} + \mathbf{b}\}$ a dependent set of vectors?
   iii) Is $\{\mathbf{a} + \mathbf{b}, 2\mathbf{a} + 2\mathbf{b}\}$ a dependent set of vectors?

2. Let **a** and **b** be as in Question 1.
   Find a relationship between m and n so that $\{2\mathbf{a} + 3\mathbf{b}, m\mathbf{a} + n\mathbf{b}\}$ is a dependent set of vectors.
3. Again, let **a**,**b** be as in Question 1.
   Is $\{\mathbf{a} + 2\mathbf{b}, 2\mathbf{a} + \mathbf{b}, 3\mathbf{a}\}$ a dependent set of three vectors? If so, write down a linear combination of two of them which equals the third.

4. Repeat Question 3 for the set $\{2\mathbf{a} - 3\mathbf{b}, 3\mathbf{a} + 4\mathbf{b}, 4\mathbf{a} - 5\mathbf{b}\}$.
5. Is $\{m\mathbf{a} + n\mathbf{b}, p\mathbf{a} + q\mathbf{b}, r\mathbf{a} - s\mathbf{b}\}$ a dependent set?

Exercise 4.1 Answers

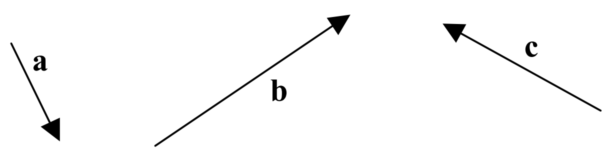
1. i) No. ii) No iii) Yes
2. $\frac{m}{n} = \frac{2}{3}$
3. Yes $\quad \mathbf{a} + 2\mathbf{b} = 2(2\mathbf{a} + \mathbf{b}) - 3\mathbf{a}$
4. Yes $\quad 2\mathbf{a} - 3\mathbf{b} = -\frac{2}{31}(3\mathbf{a} + 4\mathbf{b}) + \frac{17}{31}(4\mathbf{a} - 5\mathbf{b})$
5. Yes

Theorem 4.2

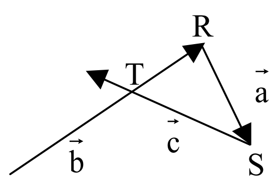
'Any three coplanar vectors are linearly dependent'

Proof

Let vectors be



Translate $\vec{b}$ and $\vec{c}$ so that a triangle RST is formed as shown in the diagram.



Now $\overrightarrow{RT}$ is a multiple of $\vec{b}$ and $\overrightarrow{TS}$ is a multiple of $\vec{c}$.

i.e. let $\overrightarrow{RT} = m\vec{b}$ and $\overrightarrow{TS} = n\vec{c}$

But $\vec{a}$ is a linear combination of $\vec{b}$ and $\vec{c}$

i.e. $\vec{a}$ is a linear combination of $\vec{b}$ and $\vec{c}$.

i.e. $\{\vec{a}, \vec{b}, \vec{c}\}$ is a dependent set.

This is a very important theorem, i.e. converse of theorem 4.1 and will be used extensively.

Three vectors are coplanar $\Leftrightarrow$ three vectors are dependent

4.2 <u>Definition</u> <u>Linear Independence</u>

Any set of two or three vectors which is **<u>not linearly dependent</u>** is said to be **<u>linearly independent</u>**.

It follows from theorem 4.2 that:

Three linearly independent vectors are not coplanar.

From here on, linearly independent will be abbreviated to l.i.

Two l.i. vectors $\Leftrightarrow$ two non-parallel vectors. Three l.i. vectors $\Leftrightarrow$ three non-coplanar vectors.

<u>Example</u>

Let $\{\mathbf{a}, \mathbf{b}\}$ be a l.i. set. Show $\{\mathbf{a} + \mathbf{b}, \mathbf{a} - \mathbf{b}\}$ is a l.i. set.

<u>Proof</u> (by contradiction)

<u>Assume</u> $\{\mathbf{a} + \mathbf{b}, \mathbf{a} - \mathbf{b}\}$ is a **<u>dependent</u>** set.

i.e. $\mathbf{a} + \mathbf{b} = m(\mathbf{a} - \mathbf{b})$for some scalar m

i.e. $\mathbf{b} + m\mathbf{b} = m\mathbf{a} - \mathbf{a}$

$(1+m)\mathbf{b} = (m - 1)\mathbf{a}$

i.e. $\mathbf{b}$ is a multiple of $\mathbf{a}$

i.e. $\mathbf{b}$ is parallel to $\mathbf{a}$ which contradicts the fact that $\{\mathbf{a}, \mathbf{b}\}$ is a l.i. set.

$\therefore$ $\{\mathbf{a} + \mathbf{b}, \mathbf{a} - \mathbf{b}\}$ is a l.i. set.

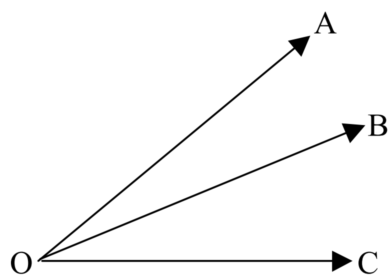
■

<u>Example</u>

Let {**i**, **j**, **k**}be the usual l.i. set. Find m so that POINTS O,A,B,C, are coplanar where **OA** = m**i** + **j**, **OB** = **I** + m**j** + **k**, **OC** = -3**i** + **k**.

Before the solution to this question is given, the reader should note that one of the real advantages of linear algebra is that it is not necessary to be able to visualise this type of question in order to answer it. Knowledge of the linear algebra concepts involved is quite sufficient.

<u>Solution</u>



O,A,B,C are coplanar points. Therefore in particular $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{OC}$ are coplanar vectors. (There is no special significance to this choice of three vectors, any three vectors will do, provided they make use of all the coplanar points.)

$\therefore \left\{\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{OC}\right\}$ is a dependent set (Theorem 4.2)

i.e. $\overrightarrow{AB} = r\overrightarrow{BC} + s\overrightarrow{OC}$ for some scalars r and s.

$$\left\{\begin{array}{l} \text{But } \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} \\ \text{and } \overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} \end{array}\right\}$$

$$\therefore\ (\vec{i} + m\vec{j} + \vec{k}) - (m\vec{i} + \vec{j}) = r\left\{(-3\vec{i} + \vec{k}) - (\vec{i} + m\vec{j} + \vec{k})\right\} + s(-3\vec{i} + \vec{k})$$

i.e. $\therefore (1 - m)\vec{i} + (m - 1)\vec{j} + \vec{k} = (-4r - 3s)\vec{i} - rm\vec{j} + s\vec{k}$

Clearly **<u>a</u>** solution exists by equating the scalars of $\vec{i}, \vec{j}, \vec{k}$ on both sides of the equation.⊗

Therefore let $1 - m = -4r - s$ (scalar multiple of $\vec{i}$)

$m - 1 = -rm$ (scalar multiple of $\vec{j}$)

$1 = s$ (scalar multiple of $\vec{k}$)

Solving these equations yields m = ± 2.

In fact these are the **only** solutions for m. We shall show this shortly.

Exercise 4.2

Let $\{\vec{i},\vec{j},\vec{k}\}$ be the usual l.i. set.

1. $$\begin{aligned}\vec{a} &= 3\vec{i}+4\vec{j}-2\vec{k}\\ \vec{b} &= 5\vec{i}-2\vec{j}+3\vec{k}\end{aligned}$$

   Find $\vec{a}-\vec{b}$ in terms of $\vec{i},\vec{j},\vec{k}$

2. True or False?

   i) $\{\vec{a},\vec{b},3\vec{a}-\vec{b}\}$ is a dependent set

   ii) $\{\vec{i},\vec{j},\vec{k}\}$ is a l.i. set

   iii) $|\mathbf{i}+\mathbf{j}| = |\mathbf{j}+\mathbf{k}|$

   iv) $|\mathbf{i}+\mathbf{j}| = 2$

   v) $|\mathbf{i}+\mathbf{j}| = \sqrt{2}$

   vi) $|\mathbf{i}+\mathbf{j}+\mathbf{k}| = \sqrt{3}$

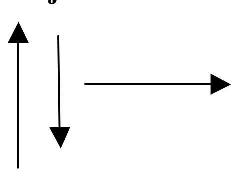
   vii) $\{\mathbf{a},\mathbf{b}\}$ is a l.i. set $\rightarrow$ $\{\mathbf{a}+2\mathbf{b}, 2\mathbf{a}-\mathbf{b}\}$ is a l.i. set.

   viii) $|\mathbf{i}+\mathbf{j}| + 2 > |\mathbf{j}+2\mathbf{k}| + 1$

   ix) $|\mathbf{i}+\mathbf{j}| > |\mathbf{i}-\mathbf{j}|$

   x) $\mathbf{i}+\mathbf{j} > \mathbf{i}-\mathbf{j}$

   xi) $\mathbf{i}+\mathbf{j}+\mathbf{k}$ is well-defined.

   xii)  is a dependent set of vectors.

   xiii) $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ is a l.i. set $\rightarrow$ $\{\mathbf{a},\mathbf{b},n\mathbf{c}\}$ is a l.i. set for all $n \neq 0$.

   xiv) $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ is a dependent set $\rightarrow$ $\{\mathbf{a},\mathbf{b},n\mathbf{c}\}$ is a dependent set for all n.

3. Given parallelogram ABCD with E as mid-point of AB such that DE intersects AC at F. Show $\mathbf{AF} = \frac{1}{3}\mathbf{AC}$.
4. Is $\{\mathbf{i} + \mathbf{j}, \mathbf{j} + \mathbf{k}, \mathbf{i} + \mathbf{k}\}$ a l.i. set?
5. Find m so that $\{m\mathbf{i} + \mathbf{2j} + 5\mathbf{k}, 2\mathbf{i} - \mathbf{j} + \mathbf{k}, \mathbf{i} + \mathbf{j} - 4\mathbf{k}\}$ is a dependent set.
6. Explain in one sentence why $2\mathbf{a} + 7\mathbf{b}$, $11\mathbf{a} - 5\mathbf{b}$, $17\mathbf{a} + 8\mathbf{b}$ are three coplanar vectors.
7. Let $\{\mathbf{a},\mathbf{b}\}$ be a l.i. set. Let $\mathbf{c} = m\mathbf{a} + n\mathbf{b}$ for some fixed non-zero scalars m and n. Investigate what type of geometric object the set V is, where

   $V = \{r\mathbf{a} + s\mathbf{b} + t\mathbf{c} \mid \text{for all } r,s,t \in \text{Reals}\}$

   i) Is V a point, circle, cube, line, sphere, plane, etc?

   ii) Is $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ a l.i. set?

   iii) Is $\mathbf{c}$ in the same plane as $\mathbf{a}$ and $\mathbf{b}$?
8. $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ is a dependent set.

   $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{c} = \mathbf{i} + 3\mathbf{j} + m\mathbf{k}$.

   Find the value of m.
9. $\{\mathbf{a},\mathbf{b}\}$ is a l.i. set. Prove $\{\mathbf{a} + 2\mathbf{b}. 2\mathbf{a} + \mathbf{b}\}$ is a l.i. set.
10. Find m so that $\mathbf{i} + \mathbf{j}, \mathbf{j} + \mathbf{k}, m\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ are three coplanar vectors.
11. $\mathbf{OA} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{OB} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{OC} = \mathbf{i} + 2\mathbf{k}$, $\mathbf{OD} = m\mathbf{j} + \mathbf{k}$. Points A,B,C,D are coplanar. Find m

<u>Exercise 4.2 Answers</u>

1. $-2\mathbf{i} + 6\mathbf{j} - 5\mathbf{k}$
2. i) T ii) T iii) T iv) F v) T vi) T vii) T viii) T ix)F x) F xi) T xii) T xiii) T xiv) T
4. Yes.
5. m = -11
6. They all lie in the plane containing **a** and **b**.
7. i) plane ii) No iii) Yes
8. m = 4.
10. m = 3.
11. m = 2.

4.3 Any set of vectors containing the zero vector is dependent because $\mathbf{0}$ = zero times any vector is trivially a linear combination of any other set of vectors.

Theorem 4.3

If $\mathbf{b}$ and $\mathbf{c}$ are two l.i. vectors and if $\mathbf{a} = m\mathbf{b} + n\mathbf{c}$ then that representation of $\mathbf{a}$ is unique.

Proof

Let $\mathbf{a} = m\mathbf{b} + n\mathbf{c}$ and $\mathbf{a} = p\mathbf{b} + q\mathbf{c}$

i.e. $p\mathbf{b} + q\mathbf{c} = m\mathbf{b} + n\mathbf{c}$

i.e. $(p - m)\mathbf{b} = (n - q)\mathbf{c}$

But since $\mathbf{b}$,$\mathbf{c}$ are l.i. vectors it follows that the only way this is true occurs when $p - m = n - q = 0$.

i.e. $p = m$ and $n = q$

i.e. the representation is unique. ■

Note:- This only applies if $\mathbf{b}$ and $\mathbf{c}$ are l.i. For example, trivially if $\mathbf{b} = \mathbf{c}$ then $\mathbf{a} = 3\mathbf{b} + 4\mathbf{c} = 2\mathbf{b} + 5\mathbf{c}$ of course and representation is not unique.

Theorem 4.4

A set of vectors is l.i. if and only if the only linear combination of those vectors yielding the zero vector occurs when the scalars are individually zero.

$\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ is l.i. iff ($m\mathbf{a} + n\mathbf{b} + p\mathbf{c} = \mathbf{0}$ implies $m = n = p = 0$)

Proof (by contradiction) let $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ be a l.i. set.

Assume $m\mathbf{a} + n\mathbf{b} + p\mathbf{c} = \mathbf{0}$ and assume, without loss of generality that $n \neq 0$ and $p \neq 0$ (if only one scalar is non-zero we have a trivial contradiction.)

Then $n\mathbf{b} = -m\mathbf{a} - p\mathbf{c}$

i.e. $\mathbf{b} = \frac{-m}{n}\mathbf{a} - \frac{p}{n}\mathbf{c}$

Now clearly **b** is a linear combination of **a** and **c** and hence $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ is a dependent set – contradiction.

■

This is the standard method for investigating the linear independence or otherwise of a set of vectors. It applies for a set of any number of vectors (finite or infinite) but sets of more than three vectors are beyond the scope of this book.

Example

Examine whether $\{\mathbf{i}+\mathbf{j}+\mathbf{k},\ \mathbf{i}-\mathbf{j}-\mathbf{k},\ 3\mathbf{i}-\mathbf{j}-\mathbf{k}\}$ is a l.i. set.

Let $m(\mathbf{i}+\mathbf{j}+\mathbf{k})+n(\mathbf{i}-\mathbf{j}-\mathbf{k})+p(3\mathbf{i}-\mathbf{j}-\mathbf{k})=\mathbf{0}$ and investigate whether m,n,p have to be zero (see Theorem 4.4)

Rearranging, we get:

$$(m+n+3p)\mathbf{i}+(m-n-p)\mathbf{j}+(m-n-p)\mathbf{k}=\mathbf{0}$$

But $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ is a l.i. set. By Theorem 4.4

$$\left.\begin{aligned} m+n+3p&=0 \quad ① \\ m-n-p&=0 \quad ② \\ m-n-p&=0 \quad ③ \end{aligned}\right\}$$

Solving this yields $m = -p$

$n = -2p$

Now note that this does <u>not</u> <u>guarantee</u> that $m = n = p = 0$ (clearly m,n,p <u>could</u> be zero but that is not the point. The point is "do they <u>have to be zero</u>".)

$m = 2,\ n = 4,\ p = -2$ is a solution for example. Therefore the set is <u>DEPENDENT</u>.

Example

Examine whether $\{\mathbf{i}+\mathbf{j}+\mathbf{k},\ 2\mathbf{i}+\mathbf{j}-\mathbf{k},\ 3\mathbf{j}-3\mathbf{k}\}$ is a l.i. set.

Let $m(\mathbf{i}+\mathbf{j}+\mathbf{k})+n(2\mathbf{i}+\mathbf{j}-\mathbf{k})+p(3\mathbf{j}-3\mathbf{k})=\mathbf{0}$.

i.e $(m+2n)\mathbf{i}+(m+n+3p)\mathbf{j}+(m-n-3p)\mathbf{k}=\mathbf{0}$.

Since $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ is a l.i. set

$$\begin{aligned} m+2n&=0 \quad ① \\ m+n+3p&=0 \quad ② \\ m-n-3p&=0 \quad ③ \end{aligned}$$

Adding ② and ③ yields $2m = 0$

$m = 0$

Substituting in ① gives $n = 0$

$p = 0.$

i.e. $n = n = p = 0$ is the <u>ONLY</u> solution.

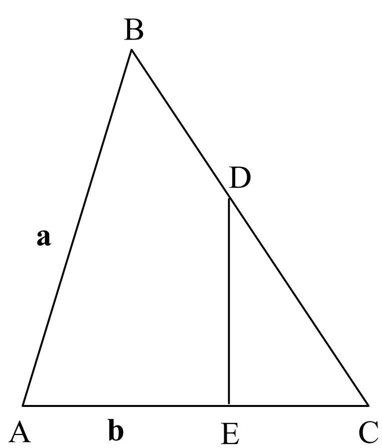
$\therefore$ { **i**+ **j** + **k**, 2**i** + **j** – **k**, 3**j** – 3**k** }is a l.i. set.

<u>Exercise 4.3</u>

1. If **x** = m**a** + n**b** and **x** = 2**a** + 3**b**, does it follow that m = 2 and n = 3? Does it make any difference if you know {**a**,**b**} is a l.i. set? Are **x**,**a**,**b** coplanar?
2. i) If m**i** + n**j** = 4**j** does n = 4 necessarily?

   ii) Find a relationship between m,n,p,q so that {m**i** + n**j** , p**i** + q**j**} is a dependent set.
3. Investigate whether {**i** + **j** + **k**, **i** + **j**, **i**} is a l.i. set.
4. **OA** = **i + k**, **OB** = 2**i** + **j** + 2**k**, **OC** = 2**i** – **j** + 2**k**, **OD** = m**i** + 3**k**

   Find m if A,B,C,D are coplanar. Does this value of m result in O being coplanar with A,B,C,D?
5. $\frac{3}{4}$**OP** = m**OR** + $\frac{2}{3}$**OQ**. Find m so that P,Q,R are collinear. Which point is between the other two?
6.



BD:DC = $\frac{1}{3}:\frac{2}{3}$

AE:EC = $\frac{2}{3}:\frac{1}{3}$

Let **AB** = **a**, **AC** = **b**

Express i) **DE**

ii) **EB** in terms of **a** and **b**

7. Are **i** + **j**, **i** – **j** + **k**, 2**i** + **k** coplanar vectors?
8. True or False?

   i) **a** = m**b** + n**c** → **a**,**b**,**c** are coplanar.

   ii) {**a**,**b**,**c**}is a dependent set → **c** = m**a** + n**b** for some scalar m and n.

iii) {**a**,**b**,**c**} is a l.i. set. → m**a** + n**b** + p**c** = **0** only when m = n = p = 0 .

iv) {**a**,**b**} is a dependent set → {**a**,**b**,**c**} is a dependent set

v) {**a**,**b**,**c**} is a dependent set → {**a**,**b**} is a dependent set.

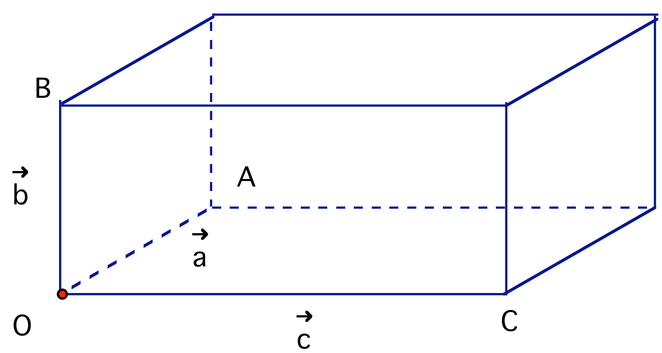
vi) {**a**,**b**} is a l.i. set → {**a**,**b**,**c**} is a l.i. set

vii) {**a**,**b**,**c**} is a l.i. set → {**a**,**b**} is a l.i. set

viii) |**a** + **b**| ≤ |**a**||**b**|

9. If **x** = m(**i** + **j**) + n(**i** – **j** + **k**) + p(2**i** + **k**) would it be possible for **x** to equal some different linear combination of {**i** + **j**, **i** – **j** + **k**, 2**i** + **k**}? See Question 7.

10. Let **a**,**b**,**c** be three vectors acting along the edges of a rectangular shoebox. $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $|\vec{c}| = 12$. Find $|\vec{a} - \vec{b} + \vec{c}|$.



11. **PQ** = m**QR** + n**RS** for some non-zero m, n. Investigate whether P,Q,R,S are coplanar.
12. {**a**,**b**} is a l.i. set. Is it possible for {**a** + **x**, **b** + **x**} to be a **<u>dependent</u>** set? Find a **<u>general</u>** expression for x in terms of **a** and **b**. If you can't do this, find a particular one (other than **x** = –**a** or – **b**)
13. {**a**,**b**} is a dependent set. Is it possible for {**a** + **x**,**b** + **x**} be l.i.? If so, give an example.
14. {**a**,**b**,**c**} is a l.i. set. Does it follow **<u>for all x</u>** that {**a** + **x**, **b** + **x**, **c** + **x**} is a l.i. set? If not, give an example or better still, a general expression for x in terms of **a**,**b**,**c**.
15. **i** + **j**, **j** + **k**, **i** + 3**j** + 2**k** are coplanar vectors (FACT). Find an **x** so that {**i** + **j** + **x**, **j** + **k** + **x**, **i** + 3**j** + 2**k** + **x**} is a l.i. set.

Exercise 4.3 Answers

1. i)No ii) Yes iii) Yes
2. i) Yes ii) $\frac{p}{q} = \frac{m}{n}$
3. Yes, it is a l.i. set
4. m = 3 Yes
5. $m = \frac{1}{12}$ R-P-Q
6. i) $\frac{1}{3}\mathbf{b} - \frac{2}{3}\mathbf{a}$ ii) $\mathbf{a} - \frac{2}{3}\mathbf{b}$
7. Yes.
8. i)True ii)False iii) True iv) True v) False vi) False vi) False vii) True viii) False
9. Yes. Since $\{\mathbf{i} + \mathbf{j}, \mathbf{i} - \mathbf{j} + \mathbf{k}, 2\mathbf{i} + \mathbf{k}\}$ is a dependent set the uniqueness theorem does not apply.
10. 13
11. They are coplanar
12. $x = \frac{1}{m-1}\mathbf{a} - \frac{m}{m-1}\mathbf{b}$ where m is any scalar $\neq 1$.
13. Yes. $\{\mathbf{i}, 2\mathbf{i}\}$ is a dependent set but if $\mathbf{x} = \mathbf{k}$ then clearly $\{\mathbf{i} + \mathbf{x}, \mathbf{2i} + \mathbf{x}\}$ is l.i.
14. No. $\mathbf{x} = \frac{m}{1-m-n}\mathbf{a} + \frac{n}{1-m-n}\mathbf{b} - \frac{1}{1-m-n}\mathbf{c}$ $(m + n \neq 1)$
15. $\mathbf{x} = \mathbf{i}$ is an example.

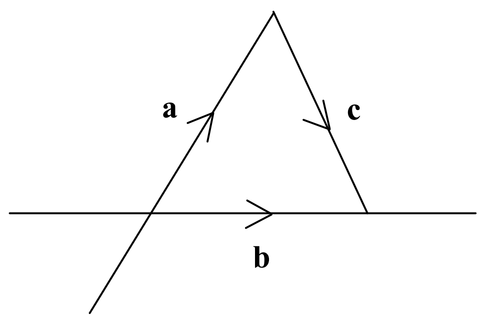
Theorem 4.5

Any vector in a plane can be written as a linear combination of any two l.i. vectors in that plane.

Proof

Let the two l.i. vectors be **a** and **b** and let **c** be an arbitrary vector in the plane of **a** and **b.** Clearly we can translate **a** so that it passes through the tail of **c** and **b** so that it passes through the tip of **c**.

i.e.



Since **a** and **b** are not parallel (they are l.i.) then they will intersect (extend **a** and **b** if necessary). Clearly **c** is then a multiple of **a** plus a multiple of **b**. i.e. linear combination of **a** and **b.** ■

We say {**a**,**b**} spans the plane. This means that any vector in the plane can be written as a linear combination of **a** and **b**.

4.4 Vector Spaces

Definition

A vector space V is a set of vectors for which the usual rules of addition and scalar multiplication are satisfied and which is:

i) closed under vector addition

ii) closed under scalar multiplication

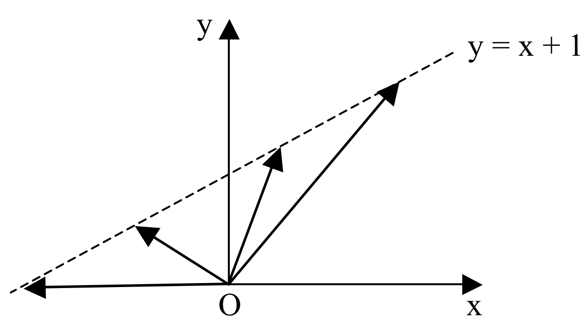
i.e. i) for any two vectors **a**,**b** in V, **a** + **b** belongs to V

ii) for any vector **a** in V and scalar m, m**a** belongs to V.

R^2 is defined to be the set of all vectors lying in a two dimensional co-ordinate system. Similarly R^3 is the set of vectors in a three dimensional coordinate system.

Example

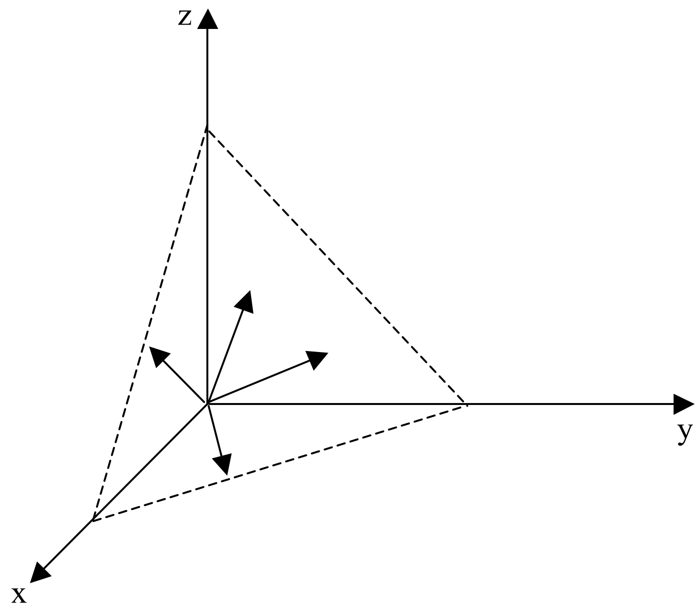
Let V be the set of <u>all</u> vectors in 2-space whose tail points are at the origin and whose tips lie on the line y = x + 1.



Then V is <u>not</u> a vector space because for example $\mathbf{j} \in V$ since **j** is the vector from (0,0) to (0,1) and (0,1) lies on the line y = x + 1, but 2**j** does not belong to V. Similarly $-\mathbf{i} \in V$ but $-\mathbf{i} + \mathbf{j} \notin V$.

<u>Example</u>

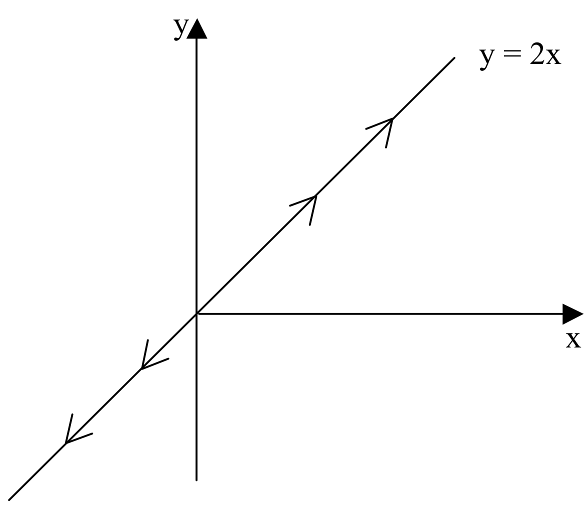
Let V be the set of vectors emanating from the origin whose tips lie on $x + y + z = 1$.



Clearly $\mathbf{i} \in V$ but $2\mathbf{i} \notin V$. Similarly $\mathbf{j} \in V$ but $\mathbf{i} + \mathbf{j} \notin V$. Hence again V is <u>not</u> a vector space.

<u>Example</u>

Let V be the set of vectors whose tips lie on the line $y = 2x$ in R^2.



Now V <u>is</u> a vector space because the conditions for a vector spare are satisfied.

It can be shown quite simply that any vector space has to contain the zero vector because if $\mathbf{a} \in V$ then $-\mathbf{a} \in V$ (-1 is a scalar) and hence $(\mathbf{a} + -\mathbf{a}) \in V$ (closed under vector addition). Therefore $\mathbf{0} \in V$ for any vector space V.

A set of vectors which is a subset of a vector space V and a vector space in its own right is called a **SUBSPACE** of V.

In fact in R^2 or R^3, a subspace is the zero vector alone, a line or plane containing the origin or R^3 itself.

Definition

Given vector space, V, a set $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ is said to span V if $\mathbf{a},\mathbf{b},\mathbf{c} \in V$ and every vector in V can be written as a linear combination of **a**,**b** and **c**.

i.e. for every **v** in V there exists m,n,p in R such that $\mathbf{v} = m\mathbf{a} + n\mathbf{b} + p\mathbf{c}$.

Example

i) $\{\mathbf{i},\mathbf{j}\}$ spans the xy plane
ii) $\{\mathbf{i},i+\mathbf{j}\}$ spans the xy plane.
iii) $\{\mathbf{i},\mathbf{j}, \mathbf{i}+\mathbf{j}\}$ spans the xy plane.
iv) $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ spans R^3 (i.e. the set of all vectors in 3-space).

Definitions

i) If a spanning set of vector space is l.i., then it is called a **BASIS** for the vector space.

   e.g. $\{\mathbf{i},\mathbf{j}\}$ is a basis for R^3

   $\{\mathbf{i},\mathbf{j}, \mathbf{i}+\mathbf{j}\}$ is **NOT** a basis for R^3

ii) The number of elements in a BASIS for a vector space is called the **DIMENSION** of that vector space.

iii) If a set of vectors is a basis for a vector space and is such that each vector has magnitude one and is perpendicular to every other vector in the basis, then the basis is called an <u>**ORTHONORMAL BASIS**</u> for the vector space.

It should be borne in mind that vector spaces are a major study in further mathematics and examples quoted here are mere simple examples. This is not intended as an exhaustive introduction but as a 'flavour' of later work.

<u>Exercise 4.4</u>

1. True or False?
   i) $\{\mathbf{i},\mathbf{k}\}$ is a basis for R^2.
   ii) $\{\mathbf{i},\mathbf{j}\}$ spans R^2.
   iii) $\{\mathbf{i},\mathbf{j}\}$ is a basis for R^2.
   iv) $\mathbf{0} \in V$ for any vector space V.
   v) $\{\mathbf{j},\mathbf{k}\}$ is a basis for the yz plane in R^3.
   vi) A basis is a l.i. set
   vii) A basis for a vector space spans the vector space.
   viii) $\{\mathbf{i},\mathbf{j}\}$ is an orthonormal basis for R^2.
   ix) $\mathbf{a},\mathbf{b} \in$ vector space $\rightarrow$ m$\mathbf{a}$ + n$\mathbf{b} \in$ vector space.

2. i) Show that $\{\mathbf{i} + \mathbf{j}, 2\mathbf{i} - \mathbf{j}\}$ is a basis for the xy plane.
   ii) Explain why $\{\mathbf{i} + \mathbf{j}, \mathbf{i} - \mathbf{j}, 2\mathbf{i}\}$ is <u>not</u> a basis for the xy plane.
   iii) Does $\{\mathbf{i} + \mathbf{j}, \mathbf{i} - \mathbf{j}, 2\mathbf{i}\}$ span the xy plane?

3. Convince yourself that: -
   i) Any two vectors in a line are dependent.
   ii) Any three vectors in a plane are dependent.
   iii) Any four vectors in 3-space are dependent.

4. i) Is $\{\mathbf{i} + \mathbf{j}, \mathbf{j} + \mathbf{k}, \mathbf{i} + \mathbf{k}\}$ a l.i. set?
   ii) Does $\{\mathbf{i} + \mathbf{j}, \mathbf{j} + \mathbf{k}, \mathbf{i} + \mathbf{k}\}$ span R^3.
   iii) Is $\{\mathbf{i} + \mathbf{j}, \mathbf{j} + \mathbf{k}, \mathbf{i} + \mathbf{k}\}$ a basis for R^3?

5. Repeat Question 4 using {**i** + **j**, **j** + **k**, **i** – **k**}
6. Convince yourself that: -
   i) Any one l.i. vector in a vector space V of dimension 1 (i.e. a line) is a basis for that vector space V.
   ii) Any two l.i. vectors in a vector space V of dimension 2 (i.e. a plane) is a basis for V.
   iii) Any three l.i. vectors in a vector space V of dimension 3 (i.e. R^3) is a basis for V.
7. Convince yourself that: -
   i) Any 2 vectors which span R^2 are l.i. and hence a basis for R^2.
   ii) Any 3 vectors which span R^3 are l.i. and hence a basis for R^3.

8. Prove that if V_1 and V_2 are vector spaces then their intersection is also a vector space. Is the union of the two vector spaces a vector space?

<u>Exercise 4.4 Answers</u>

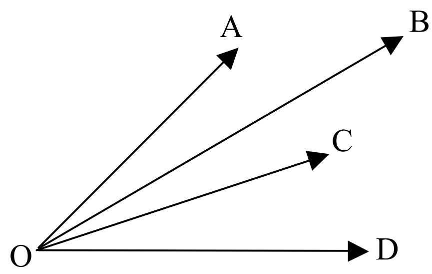
1.	i) False	ii) True	iii) True	iv) True	v) True
	vi)True	vii) True	viii) True	ix) True	x) True
2.	ii) The set is dependent.		iii) yes		
4.	i) Yes.	ii) Yes	iii) Yes.		
5.	i) No.	ii) No	iii) No.		

<u>Example</u>

Let four points A,B,C,D be <u>coplanar</u>. Let O be any external point and let **OA** = **a**, **OB** = **b**, **OC** = **c** and **OD** = **d** (If O is the origin we usually call **a** the **<u>POSITION VECTOR</u>** of A).

Then m**a** + n**b** + p**c** + q**d** = **0** where m + n + p + q = 0

Proof



A,B,C,D are coplanar

$\therefore$ **AB**, **BC**, **CD** are coplanar

$\therefore$ **CD** = r**AB** + s**BC** where r and s are scalars.

$\therefore \mathbf{d} - \mathbf{c} = r(\mathbf{b} - \mathbf{a}) + s(\mathbf{c} - \mathbf{b})$

i.e. $-r\mathbf{a} + (r - s)\mathbf{b} + (1 + s)\mathbf{c} + (-1)\mathbf{d} = \mathbf{0}$

Now note that the sum of the scalars is zero.

i.e. by letting $m = -r$, $n = r - s$, $p = 1 + s$, $q = -1$

we have $m + n + p + q = 0$ ■

Compare with Exercise 1.5 questions 7 and 9.

Note the patterns established on Exercise 4.4 # 3, 6, 7.

Similarly note that theorem 4.4 on linear independence can be extended to any number of vectors. The point to note is that a pattern can be established and mathematicians use patterns like these to investigate problems in n dimensional space which of course cannot be visualized. Try to note other patterns. For example, the fact that perpendicular lines have slopes which are negative reciprocals is truly merely a simple version of an idea which can be applied to any dimension. The simple idea is that the dot product of two vectors equalling zero is an indication of perpendicularity.

Example

Is {$\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$, $\mathbf{i} + \mathbf{j} + \mathbf{k}$}a basis for R^3.
We need to see whether i) the set is l.i. ii) the set spans R^3.

Part I To see whether {$\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$, $\mathbf{i} + \mathbf{j} + \mathbf{k}$} is l.i.

We let m($\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$) + n(3$\mathbf{k}$, $2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$) + p(3$\mathbf{k}$, $\mathbf{i} + \mathbf{j} + \mathbf{k}$) = $\mathbf{0}$ and investigate whether m,n,p individually have to be zero.
Rearranging the left-hand side we get:

$$(m + 2n + p)\mathbf{i} + (2m + n + p)\mathbf{j} + (3m + 3n + p)\mathbf{k} = \mathbf{0}$$

But since {$\mathbf{i}$,$\mathbf{j}$,$\mathbf{k}$} is a l.i. set we know

$$m + 2n + p = 0$$
$$2m + n + p = 0$$
$$3m + 3n + p = 0$$

Solving these three equations by elementary algebra methods produce the result that $m = n = p = 0$ is the only solution. It therefore follows that:

{$\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$, $\mathbf{i} + \mathbf{j} + \mathbf{k}$} is a l.i. set.

Part II

Since {$\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$, $\mathbf{i} + \mathbf{j} + \mathbf{k}$}is a set of three l.i. vectors, it follows that the three vectors are not coplanar. It therefore follows that they span R^3.

Therefore the set {$\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$, $\mathbf{i} + \mathbf{j} + \mathbf{k}$}is a basis for R^3.

Chapter Review Exercise

1. $\mathbf{AB} = \mathbf{j} + 2\mathbf{k}$, $\mathbf{AC} = 2\mathbf{i} + \mathbf{k}$, $\mathbf{AD} = \mathbf{i} + 2\mathbf{j}$. Are A,B,C,D, coplanar?
2. $\{2\mathbf{i} - \mathbf{j} + \mathbf{k}, \mathbf{i} + \mathbf{j} - \mathbf{k}, 2\mathbf{i} + 3\mathbf{j} + m\mathbf{k}\}$ is a dependent set of vectors. Find m.
3. i) Is $\{2\mathbf{i} + 3\mathbf{j}, 4\mathbf{j} + 2\mathbf{k}, 2\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}\}$a l.i. set?

   ii) Is $\{2\mathbf{i} + 3\mathbf{j}, 4\mathbf{j} + 2\mathbf{k}, 2\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}\}$a basis for R^3?

   iii) Does$\{2\mathbf{i} + 3\mathbf{j}, 4\mathbf{j} + 2\mathbf{k}, 2\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}\}$span R^3.
4. Show that $\{2\mathbf{i} + \mathbf{k}, \mathbf{i} + 2\mathbf{j}, \mathbf{j}\}$is a basis for R^3.
5. $\{\mathbf{a},\mathbf{b}\}$ is a l.i set and $\mathbf{c} = m\mathbf{a} + n\mathbf{b}$ for some fixed non-zero values of m and n.

   i) Describe geometrically the set $V = \{r\mathbf{a} + s\mathbf{b} + t\mathbf{c} \mid r,s,t \in R\}$

   ii) Is $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$a basis for V?

   iii) Does $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$span V?
6. $\mathbf{AB} = m\mathbf{i} + \mathbf{j}$, $\mathbf{AC} = \mathbf{i} + m\mathbf{j} + \mathbf{k}$, $\mathbf{AD} = -15\mathbf{i} + \mathbf{k}$.

   Find m so that A,B,C,D are coplanar.
7. $\mathbf{OA} = \mathbf{i} + \mathbf{k}$, $\mathbf{OB} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{OC} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{OD} = m\mathbf{i} - 3\mathbf{k}$.

   Find m so that A,B,C,D are coplanar. Is O coplanar with them?
8. Let V $\{r(\mathbf{i} - \mathbf{j}) + s(\mathbf{j} + \mathbf{k}) + t(\mathbf{i} + \mathbf{k}) \mid r,s,t \in \text{Reals}\}$.

   State dimension of V and find a basis for V.
9. Investigate whether $\{\mathbf{i} + \mathbf{j}, 2\mathbf{i} - \mathbf{j} - \mathbf{k}, 2\mathbf{j} + 3\mathbf{k}\}$ is a basis for R^3.
10. Show that the mid-points of two pairs of opposite edges of a tetrahedron are coplanar.
11. $\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = \mathbf{b}$, $\mathbf{OC} = \mathbf{c}$, $\mathbf{OD} = 2\mathbf{a} + 2\mathbf{b} - 3\mathbf{c}$.

    Explain why or why not A,B,C,D are coplanar.
12. Find all the values of m so that $\{2\mathbf{i} + 3\mathbf{j}, 6\mathbf{i} + m\mathbf{j} + m\mathbf{k}, 3\mathbf{j} + 4\mathbf{k}\}$ is a dependent set of vectors.

Chapter Review Exercise Answers

1. No. 2. m = -3. 3. i) No ii) No iii) No.

5. i) A plane of vectors ii) No iii) No iv) Yes.

6. $m = \pm 4$. 7. m = 5 No. 8. Dimension V = 2. $\{\mathbf{i} - \mathbf{j}, \mathbf{j} + \mathbf{k}\}$

9. Yes it is. 11. Points A,B,C,D are coplanar. 12. m = 36

Practice Test 1 on Chapter 4

1. True or False?
    a) $|\mathbf{i} + \mathbf{j}| = 2$
    b) If $\mathbf{AB} = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{BC} = 2\mathbf{i} + 4\mathbf{j}$ then A,B,C are collinear
    c) $m\mathbf{i} + n\mathbf{j} = 2\mathbf{i} + 3\mathbf{j}$ implies $m = 2$, $n = 3$.
    d) $m\mathbf{a} + n\mathbf{b} = 2\mathbf{a} + 3\mathbf{b}$ implies $m = 2$, $n = 3$.
    e) $m\mathbf{a} + n\mathbf{b} = \mathbf{0}$ implies $\{\mathbf{a},\mathbf{b}\}$ is a dependent set where m and n are non-zero
    f) $\mathbf{i} + \mathbf{j}$, $\mathbf{j} + \mathbf{k}$, $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ are coplanar vectors.
    g) $\mathbf{i} + \mathbf{j}$ is perpendicular to the z axis
    h) $\mathbf{OA} + \mathbf{OB} = \mathbf{OC}$ implies O,A,B,C are coplanar
    i) $|\mathbf{OA}| + |\mathbf{OB}| = |\mathbf{OC}|$ implies O,A,B,C are coplanar
    j) $|3\mathbf{i} + 4\mathbf{j}| = 7$
2. $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OB} = \mathbf{b}$. $\mathbf{OC} = m\mathbf{a} + n\mathbf{b}$ for some fixed value of m and n.
    a) Does it follow that O,A,B,C are coplanar?
    b) Does it make a difference whether $\mathbf{a},\mathbf{b}$ are dependent vectors?
3. $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OB} = \mathbf{b}$ and $\mathbf{OC} = \mathbf{c}$. $\mathbf{OD} = m\mathbf{a} + n\mathbf{b} + p\mathbf{c}$ for some fixed values of m,n,p.
    a) Does it follow that O,A,B,C,D are coplanar?
    b) Does it make a difference if $\mathbf{a},\mathbf{b},\mathbf{c}$ are dependent vectors?
4. Find the value(s) of m so that A,B,C,D are coplanar points where
   $\mathbf{AB} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{AC} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{AD} = m\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$
5. Is it possible to write <u>every</u> vector in R^3 as a linear combination of the vectors $\mathbf{i} + \mathbf{j}$, $\mathbf{j} + \mathbf{k}$, and $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$. Answer yes or no with an explanation.
6. Is $\{\mathbf{i} + \mathbf{j}, \mathbf{i} + 2\mathbf{j}\}$ a basis for R^2. Explain.

Practice Test 1 on Chapter 4 Answers

1. a) F b) T c) T d) F e)T f)T g)T h)T i)F j)F
2. a)Yes b)No 3. a) No b) Yes then the points <u>are</u> coplanar
4. 32 5. No because the three vectors are coplanar
6. Yes because they are two independent (i.e. non-parallel) vectors in R^2.

Practice Test 2 on Chapter 4

1. True or False?
   a) If $|\mathbf{a}| = 3$ and $|\mathbf{b}| = 4$ then $|\mathbf{a} + \mathbf{b}| = 7$
   b) $\{\mathbf{i} + \mathbf{i} + \mathbf{j}, + 2\mathbf{i} + \mathbf{j}\}$is a dependent set
   c) $3\mathbf{i} + 4\mathbf{j}$ is parallel to $6\mathbf{i} + 8\mathbf{j}$
   d) $\mathbf{i},\mathbf{j},\mathbf{k}$ are coplanar
   e) Any three vectors in a plane form a dependent set.
   f) Any three vectors in R^2 form a dependent set
   g) If $m\mathbf{a} = n\mathbf{b}$ and $\{\mathbf{a},\mathbf{b}\}$ is an independent set then $m = n = 0$.
   h) $\mathbf{i} + \mathbf{j}$, $\mathbf{j} + \mathbf{k}$, $\mathbf{i} + \mathbf{k}$ lie in a plane.
   i) Any set of three vectors which span R^3 is a basis for R^3.
   j) Any set of four vectors in R^3 is a dependent set.
   k) Any set of three vectors in R^2 is a dependent set.
   l) If $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$is a dependent set then $\{\mathbf{a},\mathbf{b}\}$ is a dependent set
   m) If $\{\mathbf{a},\mathbf{b}\}$ is a dependent set then $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$is a dependent set
   n) If $\{\mathbf{a},\mathbf{b}\}$is an independent set then $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$is an independent set
   o) If $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ is an independent set then $\{\mathbf{a},\mathbf{b}\}$is an independent set
2. $\mathbf{AB} = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{BC} = 2\mathbf{i} + 4\mathbf{j}$. Are A,B,C necessarily collinear?
3. Investigate whether $\{\mathbf{i} - \mathbf{j} + \mathbf{k}, \mathbf{i} + \mathbf{j} - \mathbf{k}, -\mathbf{i} + \mathbf{j} + \mathbf{k}\}$ is an independent set or not.
4. A,B,C,D are four coplanar points. O is the origin.
   $\mathbf{OA} = 3\mathbf{i}$, $\mathbf{OB} = 3\mathbf{j}$ and $\mathbf{OC} = 3\mathbf{k}$. $\mathbf{OD} = m\mathbf{i} + m\mathbf{j} + 2m\mathbf{k}$. Find the value(s) of m.
5. Draw a nice large diagram where $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} + \mathbf{c}|$ but $\mathbf{b} = -2\mathbf{c}$

Practice Test 2 on Chapter 4 Answers

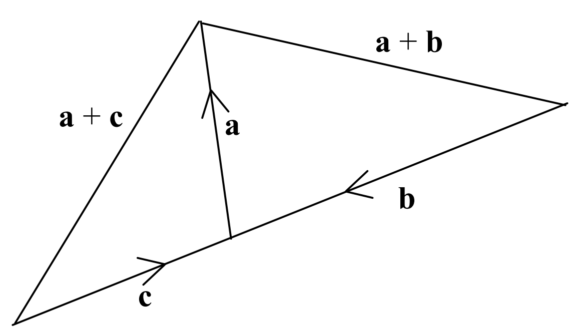
1. a)F b)T c)T d)F e)T f)T g)T h)F i)T j)T k)T l)F
   m)T n)F o)T

2. Yes

3. Yes they are independent

4. $m = \dfrac{3}{4}$

5.



Practice Test 3 on Chapter 4

1. True or False?
   a) $|\mathbf{i} + \mathbf{j} + \mathbf{k}| = \sqrt{3}$
   b) $\mathbf{i} + \mathbf{j} > \mathbf{i}$
   c) $|\mathbf{i} + \mathbf{j}| > |\mathbf{i}|$
   d) $|\mathbf{i} + \mathbf{j}| = |\mathbf{j} + \mathbf{k}|$
   e) $2\mathbf{i} + 3\mathbf{j} = m\mathbf{i} + n\mathbf{j} \Rightarrow m = 2$ and $n = 3$.
   f) $|3\mathbf{i} + 4\mathbf{j}| = 7$
   g) $|3\mathbf{i}| + |4\mathbf{j}| = 7$
   h) $|\mathbf{i} + \mathbf{j}| = \sqrt{2}$
   i) $m\mathbf{a} + n\mathbf{b} = 0 \Rightarrow \{\mathbf{a},\mathbf{b}\}$ is a dependent set assuming m, n are non-zero
   j) $2\mathbf{i} + 3\mathbf{j}$, $3\mathbf{i} + 4\mathbf{j}$ are dependent vectors
   k) $2\mathbf{i} + 3\mathbf{j}$, $3\mathbf{i} + 3\mathbf{k}$ are dependent vectors
   l) If $\mathbf{AB} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{BC} = 3\mathbf{i} + 4\mathbf{j}$ then A,B,C are collinear points.
   m) $\mathbf{i}$ is perpendicular to $\mathbf{j} + \mathbf{k}$
   n) $\{\mathbf{i}, \mathbf{j}, \mathbf{j}+\mathbf{k}\}$ is an independent set of vectors.
2. Draw a diagram to illustrate the falsity of the implication

   $$|\mathbf{a} + \mathbf{b}| = |\mathbf{a} + \mathbf{c}| \Rightarrow |\mathbf{b}| = |\mathbf{c}|$$
3. $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ is a set of three independent vectors. Is it possible for $\{m\mathbf{a} + n\mathbf{b}, \mathbf{b}, \mathbf{c}\}$ to be a dependent set except in the trivial cases?
4. Find the value(s) of m so that $\{m\mathbf{i} + 2\mathbf{j}, 3\mathbf{j} + 4\mathbf{k}, \mathbf{i} + 3\mathbf{k}\}$ is a dependent set.
5. Are $\mathbf{i} + \mathbf{j}$, $\mathbf{i} - \mathbf{j} + \mathbf{k}$, $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ coplanar vectors? Explain
6. A,B,C and D are four coplanar points. Let O be the origin (0,0,0) in R^3 (three dimensional space)

   $\mathbf{OA} = \mathbf{i}$, $\mathbf{OB} = \mathbf{j}$, $\mathbf{OC} = \mathbf{k}$, $\mathbf{OD} = m\mathbf{i} + m\mathbf{j} + m\mathbf{k}$. Find the value(s) of m.

Answers

1. a) T b) F c)T d) T e)T f)F g)T h)T i)T j)F k)F l)F m)T n)T

2. --- 3. No 4. $m = -\dfrac{8}{9}$

5. Yes since $3\mathbf{i} - \mathbf{j} + \mathbf{k} = 1(\mathbf{i} + j + 2(\mathbf{i} - \mathbf{j} + \mathbf{k})$

6. $m = \dfrac{1}{3}$

Practice Test 4 on Chapter 4

1. True or False.
   a) $\{\mathbf{a},\mathbf{b}\}$ is l.i. $\Rightarrow$ $\{\mathbf{a} + 2\mathbf{b}, 2\mathbf{a} + \mathbf{b}\}$ is l.i.
   b) $\{\mathbf{a} + 2\mathbf{b}, 2\mathbf{a} + \mathbf{b}\}$ is l.i. $\Rightarrow$ $\{\mathbf{a},\mathbf{b}\}$ is l.i.
   c) $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ is l.i. $\Rightarrow$ $\mathbf{c} = m\mathbf{a} + n\mathbf{b}$ is false for **all** m and n.
   d) If ($|\mathbf{a}| = 3$ and $|\mathbf{b}| = 4$) then ($|\mathbf{a} + \mathbf{b}| = 7 \Rightarrow \{\mathbf{a},\mathbf{b}\}$ is a dependent set)
   e) If $\mathbf{AB} = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{BC} = 2\mathbf{i} + 4\mathbf{j}$ then A-B-C are collinear points.
   f) $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ is l.i. $\Rightarrow$ $\{\mathbf{a},\mathbf{b}\}$ is a l.i. set
   g) $m\mathbf{i} + n\mathbf{j} + p\mathbf{k} = \mathbf{0} \Rightarrow m = n = p = 0$
   h) Any three vectors in a plane form a dependent set.
   i) $\mathbf{i} + \mathbf{j}$, $\mathbf{j} + \mathbf{k}$, $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ are coplanar vectors.
   j) $\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$ is a vector of magnitude 1.
   k) (It is Sunday $\Rightarrow$ liquor stores are closed) $\Leftrightarrow$ (Liquor stores are open $\Rightarrow$ It is not Sunday)
   l) $\mathbf{PQ} = 2\mathbf{QR} + 3\mathbf{RS} \Rightarrow$ P,Q,R,S are necessarily coplanar.

2. Find all the value(s) of m so that $\{m\mathbf{i}+ \mathbf{j} + \mathbf{k}, \mathbf{i} + m\mathbf{j} + \mathbf{k}, \mathbf{i} + \mathbf{j} + m\mathbf{k}\}$ is a dependent set of vectors.
3. O is the origin. $\mathbf{OA} = \mathbf{j} + 3\mathbf{k}$, $\mathbf{OB} = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{OC} = -2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{OD} = m\mathbf{i} + m\mathbf{j} + m\mathbf{k}$. Find those values of m so that A,B,C, D are coplanar.
4. Show whether $\{\mathbf{i}-\mathbf{j}+\mathbf{k}, \mathbf{i}+\mathbf{j}-\mathbf{k}, -\mathbf{i} + \mathbf{j}+ \mathbf{k}\}$ is a l.i. set or not

Answers

1. All of them are true
2. m = 1 or – 2
3. m can be any number
4. The set is linearly independent