Chapter 10

10. Binomial Theorem

When we multiply out (say) $(x - 1)(x - 2)(x - 3)$ we are considering all the possible terms where we are choosing one of the elements from each bracket and combining the results.
e.g. $(x - 1)(x - 2)(x - 3)$

$$= x^3 + (-1 - 2 - 3)x^2 + ((-1)(-2) + (-1)(-3) + (-2)(-3))x + (-1)(-2)(-3)$$

x^3	$+ (-1 - 2 - 3)x^2$	$+ ((-1)(-2) + (-1)(-3) + (-2)(-3))x$	$+ (-1)(-2)(-3)$
choosing x from each bracket	choosing x from two of the brackets	choosing x from one of the brackets	choosing no x's from the brackets

$= x^3 - 6x^2 + 11x - 6.$

Similarly, $(x - 1)(x - 2)(x - 3)(x - 4)$

$$= x^4 + (-10)x^3 + (+35)x^2 + (-50)x + 24.$$

Now consider

$$(x + 1)(x + 1)(x + 1)(x + 1)$$

$= x^4 + (+4)x^3 + (+6)x^2 + (+4)x + 1$

$= x^4 + 4x^3 + 6x^2 + 4x + 1.$

Note for example that + 6 is obtained by choosing + 1 from two of the 4 brackets.

i.e. $\binom{4}{2}$.

It follows that

$$(x + 1)^n = x^n + \binom{n}{1}x^{n-1} \cdot 1 + \binom{n}{2}x^{n-2} \cdot 1^2 + \ldots.1^n$$

For example

$(x + 1)^7 = x^7 + \binom{7}{1} x^6 \cdot 1 + \binom{7}{2} x^5 \cdot 1^2 + \binom{7}{3} x^4 \cdot 1^3 + \ldots . 1^7$

i.e. $(x + 1)^7 = x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^3 + 21x^2 + 7x + 1$

In general,

$$(a + b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \ldots . + b^n.$$

This is called the **BINOMIAL THEOREM**.

For example

$$(2x + 3)^4 = (2x)^4 + \binom{4}{1}(2x)^3 \cdot 3 + \binom{4}{2}(2x)^2 \cdot 3^2 + \binom{4}{3}(2x) \cdot 3^3 + 3^4$$

$\therefore (2x + 3)^4 = 16x^4 + 96x^3 + 216x^2 + 216x + 81.$

Note that in the expression since $\binom{4}{r} = \binom{4}{4-r}$ it does not matter whether the co-efficient is, for example, $\binom{4}{1}$ or $\binom{4}{3}$.

Question

Find the middle term in the expression $(3x - 4)^6$.

Answer

Middle term is $\binom{6}{3}(3x)^3(-4)^3 = -34560x^3$.

Question

Find the co-efficient of x^3 in the binomial expansion of $(2x - \frac{3}{x})^7$

Answer

The term we want is $\binom{7}{2}(2x)^5\left(\frac{-3}{x}\right)^2$

$= (21)\cdot(32x^5)\left(\frac{9}{x^2}\right) = 6048x^3.$

Question

Without using a calculator, find to the nearest \$1 the amount that \$1000 will accrue to, in 10 years, at 4% p.a. compound interest.

$$\begin{aligned}\text{Amount} &= 1000(1 + 0.04)^{10}\\ &= 1000(1^{10} + \binom{10}{1}1^9(0.04) + \binom{10}{2}1^8(0.04)^2 + \binom{10}{3}1^7(0.04)^3 + \ldots\\ &= 1000(1 + 0.4 + 45(0.0016) + 120(0.000064) + \ldots\\ &\approx 1480\end{aligned}$$

Question

Write out the binomial expansion of $(a + b + c)^4$

Answer

Remember that $(a + b + c)^4 = (a + b + c)(a + b + c)(a + b + c)(a + b + c)$ and we are choosing all possible combinations where we choose one element from each bracket.

$$\begin{aligned}\therefore (a+b+c)^4 = a^4 &+ \binom{4}{1}a^3b + \binom{4}{1}a^3c + \binom{4}{2}a^2b^2 + \binom{4}{2}a^2c^2 + \binom{4}{2}\binom{2}{1}a^2bc + \binom{4}{1}ab^3 + \binom{4}{1}ac^3\\ &+ \binom{4}{1}\binom{3}{2}ab^2c + \binom{4}{1}\binom{3}{2}abc^2 + \binom{4}{1}bc^3 + \binom{4}{1}bc^3 + \binom{4}{2}b^2c^2 + b^4 + c^4\end{aligned}$$

$$\begin{aligned}\therefore (a + b + c)^4 = a^4 &+ 4a^3b + 4a^3c + 6a^2b^2 + 6a^2c^2 + 12a^2bc + 4ab^3 + 4ac^3 + 12ab^2c\\ &+ 12abc^2 + 4bc^3 + 4b^3c + 6b^2c^2 + b^4 + c^4.\end{aligned}$$

Question

Find the value of r if the co-efficients of x^r and x^{r+1} are equal in the binomial expansion of $(3x + 2)^{19}$.

Answer

Co-efficients of the following terms $\binom{19}{r}(3x)^r(2)^{19-r}$ and $\binom{19}{r+1}(3x)^{r+1}(2)^{18-r}$ are equal.

$$\therefore \binom{19}{r}3^r \cdot 2^{19-r} = \binom{19}{r+1}3^{r+1} \cdot 2^{18-r}$$

$$\therefore \frac{19!}{r!(19-r)!}3^r \cdot 2^{19-r} = \frac{19!}{(r+1)!(18-r)!} \cdot 3^{r+1} \cdot 2^{18-r}$$

Canceling both sides by $19!\ 3^r \cdot 2^{18-r}$ we get

$$\frac{2}{r!(19-r)!} = \frac{3}{(r+1)!(18-r)!}$$

$\therefore 2(r + 1)!\ (18 - r)! = 3r!\ (19 - r)!$

$2(r + 1) = 3\ (19 - r)$

i.e. $2r + 2 = 57 \ \ 3r$

$5r = 55$

$r = 11$

$\therefore$ co-efficients of x^{11} and x^{12} are equal.

Although a proof (except for positive integers) is beyond the scope of this book the **BINOMIAL THEOREM** is true for any real power (positive, negative, fractional, irrational) subject to certain restrictions on the value of x.

For example

$$(1 + x)^{-1} = 1^{-1} + (-1)1^{-3}x + \frac{(-1)(-2)}{1\times 2}1^{-3}x^2 + ...$$

$$= 1 - x + x^2 - x^3 + ...$$

This is true for $-1 < x < 1$.

Note how this result conforms to the sum of an infinite geometric series.

Exercise 10.1

Binomial Theorem

1. State the middle term in the expansion of $(a + b)^{20}$.
2. Find the term not containing x in the expansion of $\left(2x + \frac{1}{x}\right)^{12}$.
3. Find the coefficient of x^{20} in the expansion of $(x^2 + 2x)^{12}$.
4. Expand $(a + b + c)^3$. (Hint $(a + (b + c))^3$).
5. Find n if the coefficients of x^3 and x^{12} are equal in the expansion of $(1 + x)^n$
6. Find an approximate value for $(1.0001)^{100}$ in your head.
7. Find the coefficient of x^{11} in the expansion of $\left(2x^2 + \frac{3}{2x}\right)^{10}$.
8. Expand $(1 + x + x^2)^3$ in ascending powers of x as far as x^3.
9. Write down the binomial expansion of $\left(y + \frac{1}{y}\right)^6$.
10. Write down the binomial expansion of $(3x - 2y)^6$.
11. Find the next term in the expansion if $1 + 12x + 54x^2$ are the first three terms.
12. Find r if the coefficients of x^r and x^{r+1} are equal in the expansion of $(3x + 2)^9$.
13. What is the sum of the coefficients of $(a + b)^{10}$?
14. What is the sum of the coefficients of $(a - b)^{10}$?
15. What is the sum of the coefficients of $(a - 2b)^{10}$?
16. For two values of x, the middle term in the expansion of $(1 + x)^{24}$ in ascending powers of x is the arithmetic mean of the term immediately before it and after it. Find those values of x.
17. What is the coefficient of $x^3y^4z^2$ in the expansion of $(x + 2y + z)^9$?
18. Find the term not containing x in the expansion of $\left(x + \frac{1}{x}\right)^3\left(x - \frac{1}{2}\right)^5$.
19. Is it possible for two consecutive terms in the expansion of $(2x + 3)^9$ to have equal coefficients? If so, find them.

Exercise 10.1 Answers

1. $\binom{20}{10}a^{10}b^{10}$
2. 59136
3. 7920
4. $a^3 + 3a^2b + 3a^2c + 3ab^2 + 3ac^2 + 6abc + b^3 + 3b^2c + 3bc^2 + c^3$
5. 15
6. 1.01
7. 51840
8. $1 + 3x + 6x^2 + 7x^3$
9. $y^6 + 6y^4 + 15y^2 + 20 + \frac{15}{y^2} + \frac{6}{y^4} + \frac{1}{y^6}$
10. $729x^6 - 1916x^5y + 4861x^4y^2 - 4320x^3y^3 + 2160x^2y^4 - 576xy^5 + 64y^6$
11. $108x^3$
12. $r = 5$
13. 1024
14. 0
15. 1
16. $\frac{2}{3}$ or $\frac{3}{2}$
17. 20160
18. $\frac{55}{16}$
19. Yes. $489888x^4$ and $489888x^3$

Exercise 10.2

Binomial Theorem Harder Questions

1. For what values of x is it possible to expand $(1 + x)^{-1}$ as a binomial expression?
2. Expand $(1 + x)^{-2}$. For what values of x is the expansion valid?
3. What is the first negative coefficient in the expansion of $(1 + x)^{3\cdot 5}$?
4. Write down the first negative coefficient in the expansion of $(1 - x^2)^{-5/3}$
5. Write $1 - x + x^2 - x^3 + x^4 - \ldots$ in a different way. For what values of x does the series converge?
6. Find the sum of $x + 2x^2 + 3x^3 + \ldots$ assuming that the sum is finite.
7. Find the largest coefficient in the expansion of $\left(2 + \frac{x}{3}\right)^9$.
8. Find the term in the expansion of $\left(2 + \frac{1}{x} + x\right)^5$ which does not contain x.
9. What is the coefficient of $x^2y^2z^2$ in the expansion of $(2x + 3y - z)^6$?
10. Show that the number of terms in the expansion of

    $(x_1 + x_2 + x_3 + \ldots\ldots x_r)^n$ is $\binom{n+r-1}{r-1}$
11. Show that $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots\ldots + \binom{n}{n}^2 = \binom{2n}{n}$

    Hint: $(1 + x)^{2n} = (1 + x)^n (1 + x)^n$

Exercise 10.2 Answers

1. $-1 < x < +1$ 2. $1 - 2x + 3x^2 - 4x^3$, $-1 < x < 1$ 3. $-\frac{7}{256}x^5$

4. $-\frac{440}{27}x^6$ 5. $\frac{1}{1+x}$ for $-1 < x < +1$ 6. $\frac{x}{(1-x)^2}$

7. 768 8. 42 9. 3240