CHAPTER EIGHT

8. Complex Numbers

When we solve $x^2 + 2x + 2 = 0$ and use the Quadratic Formula we get

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(2)}}{2(1)}$$
$$= \frac{-2 \pm \sqrt{-4}}{2}$$

Since we know that $\sqrt{-4}$ is not a real number it follows that there is no **REAL** solution to the equation $x^2 + 2x + 2 = 0$.

However, mathematicians like to investigate conjectures "I wonder what would happen if we define …. "

Consequently we define $\sqrt{-1}$ to be a "number" which we will call i (not to be confused with $\vec{i} = \overrightarrow{(1,0,0)}$. Note $i^2 = -1$.

It turns out that defining $\sqrt{-1}$ like this does not lead to contradictions in later mathematics study and in fact is extremely helpful. Numbers containing *i* in some form are called **COMPLEX NUMBERS**

Theorem

i is not a real number.

This theorem may seem self-evident but it should be remembered that there are some curious numbers out there and the fact that *i* is not real is not as obvious as it first seems.

For example

a) Is $2^{\sqrt{2}}$ real? b) Is $\lim_{x\to 0^+} \frac{1}{x}$ real? c) Is $(-2)^{3.1}$ real? d) Is $\lim_{x\to 0^+} \frac{1}{x^2}$ real?

e) Is log(-8) real? f) Is $(-8)^{\frac{1}{3}}$ real?

It all depends upon one's perspective.

For example

$$(-2) = (-8)^{\frac{1}{3}} = (-8)^{\frac{2}{6}} = ((-8^2))^{\frac{1}{6}} = (64)^{\frac{1}{6}} = +2 \text{ (a contradiction)}$$

My own opinion is that $(-8)^{\frac{1}{3}}$ is not equal to – 2 because $(-8)^{\frac{1}{3}}$ is not defined properly but many mathematicians would argue otherwise.

Back to Theorem (*i* is not real.)

We will use a contra-positive argument.

<u>Assume</u> *i* <u>is</u> real

Then *i* is zero or positive or negative.

<u>Case 1</u>

$i = 0$

Then $i \cdot i = 0 \cdot 0$

$i^2 = 0$

contradiction since $i^2 = -1$ by definition.

<u>Case 2</u>

i is positive

$\therefore\ i > 0$

$\therefore\ i \cdot i > 0$ (preserving the inequality under the assumption that *i* is <u>**POSITIVE**</u>)

$\therefore\ i^2 > 0$

$\therefore -1 > 0$

contradiction.

<u>Case 3</u>

i is negative

$\therefore\ i < 0$

$\therefore\ i \cdot i > 0 \cdot$ (reversing the inequality since i is assumed negative here)

$\therefore\ i^2 > 0$

$\therefore -1 > 0$

Since all three cases fail it follows that the original assumption is false and hence i <u>is not real.</u>

<u>Powers of i</u>

Since $i^2 = -1$

then $i^4 = +1$ and $i^3 = -i$.

It follows that powers of i can be easily obtained by considering the remainder when the power is divided by 4.

e.g. a) $i^{43} = i^{40}\, i^3 = (i^4)^{10}\, i^3 = 1^{10} \cdot i^3 = -i$

b) $i^{37} = i^{36}\, i^1 = 1 \cdot i = i$

c) $i^{46} = i^{44}\, i^2 = 1(-1) = -1$.

Note that for example

$$(1 + i)^2 = 1 + 2i + i^2 = 1 + 2i + (-1) = 2i$$

$$\sqrt{-4} = \sqrt{4}\sqrt{-1} = 2i$$

Furthermore note that when solving $x^2 + 2x + 2 = 0$ earlier we found that $x = \dfrac{-2 \pm \sqrt{-4}}{2}$

i.e. $x = \dfrac{-2+2i}{2}$ or $x = \dfrac{-2-2i}{2}$

i.e. $x = -1 + i$ or $-1 - i$

In general mathematical operations for complex numbers are as for real numbers.

e.g. $(1 + 2i) + (3 + 4i) = 4 + 6i$

$(7 - 2i) - (5 + 4i) = 2 - 6i$

$(2 + 3i)(3 + 4i) = 6 + 17i + 12i^2 = -6 + 17i$

In fact as we shall see later the set of real numbers is a subset of the set of complex numbers.

Exercise 8.1

1. Express in terms of i

   a) $\sqrt{-25}$ b) $\sqrt{-16}$ c) $\sqrt{-4}+\sqrt{-9}$ d) $\sqrt{-36}\sqrt{-4}$

2. Write in form of a + bi

   a) $(2 + 3i) + (4 + 5i)$ b) $(1 + 2i)(1 + 3i)$ c) $(1 + i)^2$ d) $(1 + i)(1 – i)$

3. Simplify

   a) i^{14} b) i^{55} c) i^{72}

4. Solve for z where z is a complex number

   $z^2 + 8z + 20 = 0$

5. Solve for z.

   $3z^2 + z + 1 = 0$

6. Simplify $(1 + i)^6$

7. Solve for z.

   $z^3 – z^2 + z – 1 = 0$

8. Solve for z.

   $z^3 – z^2 + 4z = 4$

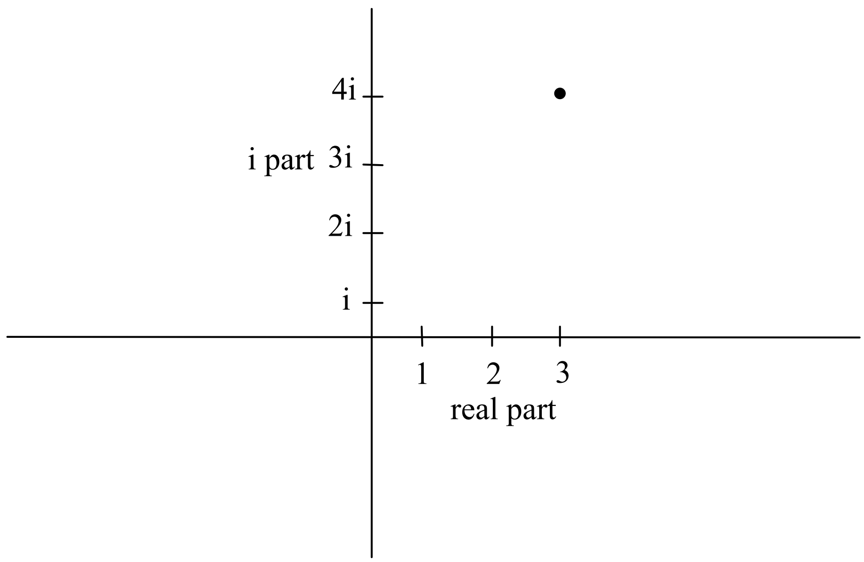
Exercise 8.1 Answers

1. a) $5i$ b) $4i$ c) $5i$ d) -12
2. a) $6 + 8i$ b) $-5 + 5i$ c) $2i$ d) 2
3. a) -1 b) $–i$ c) 1
4. $-4 + 2i$, $-4 – 2i$
5. $\dfrac{-1+\sqrt{11}i}{6}$ or $\dfrac{-1-\sqrt{11}i}{6}$
6. $-8i$
7. 1, i, $-i$
8. 1, $2i$, $-2i$

Standard Form of a Complex Number

When a complex number is written in the form a + b*i* this is called **STANDARD FORM** and it helps to represent the complex number on a diagram called an **ARGAND DIAGRAM** as shown below.

a + b*i* is represented by a dot at the position (a,b) as we understand from elementary co-ordinate geometry



Note how (say) the number 2 can be thought of as 2 + 0*i* and represented by a dot at (2,0) establishing the fact that the set of real numbers is a subset of the set of Complex Numbers.

To write $\frac{2+3i}{1+i}$ in standard form.

$$\frac{2+3i}{1+i} = \frac{(2+3i)}{(1+i)}\frac{(1-i)}{(1\text{-}i)} = \frac{2+i\text{-}3i^2}{1-i^2} = \frac{5+i}{2} = \frac{5}{2}+\frac{1}{2}i$$

i.e. $\frac{2+3i}{1+i}$ can be presented by a dot at $(\frac{5}{2},\frac{1}{2})$.

Note that a – b*i* is called the **CONJUGATE** of a + b*i*

e.g. 2 + 3*i* and 2 – 3*i* are conjugates of each other.

Complex numbers are often represented by the letter z. The conjugate of z is written $\bar{z}$.

From the Quadratic Formula it is clear that for (say) $x^2 - 4x + 13 = 0$

$$x = \frac{4+\sqrt{-36}}{2} \; or \; \frac{4-\sqrt{-36}}{2}$$

$$= 2 + 3i \; or \; 2 - 3i$$

From this it is easy to deduce for a quadratic equation with real co-efficients that if a complex number is a root then so is its conjugate.

In fact this is true for <u>any</u> polynomial equation with real co-efficients.

<u>Example</u>

Given $1 + i$ is a root of $x^3 - 26x^2 + 50x - 24 = 0$, find its three roots.

<u>Solution:</u>

Since $1 + i$ is a root then $1 - i$ is a root.

i.e. $[x - (1 + i)]\,[x - (1 - i)]\,[x - \text{other root}] = x^3 - 26x^2 + 50x - 24$

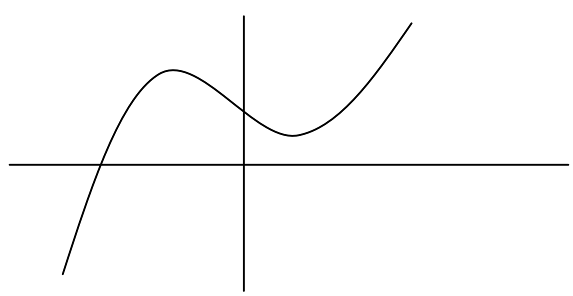
By equating constant terms on both sides of the equation we can see that

$(1 + i)(1 - i)(\text{other root}) = 24$

i.e. other root = 12

$\therefore$ roots are $1 + i$, $1 - i$ and 12.

Note also that a polynomial equation with real co-efficients whose maximum power is ODD must always have a REAL root since complex roots of such an equation always occur in pairs of conjugates. Note how this last concepts is confirmed by the fact for example that a cubic equation can be essentially presented by a graph such as that shown below which must have an x-intercept.



Furthermore two complex numbers can only be equal if their real parts and their i parts are equal separately.

For example if $a + 3i = 5 - ci$

Then $a = 5$ and $c = -3$.

Exercise 8.2

1. Show that $x^2 + 2x + 3 = 0$ has no solutions in real numbers.
2. Solve $x^2 + x + 1 = 0$ where x can be a complex number.
3. Evaluate a) $\sqrt{-9}$ b) $\sqrt{-16}$ c) $\sqrt{-9}\sqrt{-16}$ d) i^4 e) i^8 f) i^{18}
4. Simplify $(1 + i)(1 - i)$
5. Simplify $(1 + i)^8$
6. Write $\frac{1}{1+i}$ in the form $a + bi$
7. Simplify $(2 + 3i) + (3 - 4i)$
8. Express $\sqrt{-36}+\sqrt{-25}+\sqrt{-49}$ as a complex number
9. Evaluate $\frac{1+i}{1-i}$ and write its value in the form $a + bi$
10. Find all four roots of the equation $x^4 = 16$
11. Is it true that $\sqrt{a}\sqrt{b}$ equals $\sqrt{ab}$ for all numbers a and b? Does it make any difference if a and/or b is a negative number?
12. Solve $z^2 - 2z + 2 = 0$ where z is a complex number.
13. Solve $z^3 - 9z^2 + 26z - 24 = 0$ where z is a complex number.
14. If $(a + bi)^2 = -5 - 12i$ find a and b.
15. Simplify a) $\frac{1+i}{1-i}$ b) $\frac{2}{1-i}+\frac{2}{1+i}$ c) $\frac{1-i}{1+i}+\frac{1+i}{1-i}$
16. $1 + i$ is a root of the equation $z^3 - 26z^2 + 50z = 48$ (fact). Find the real roots of this equation.
17. Given that $2 + i$ is a root of $z^4 - 6z^2 + 25 = 0$ solve the equation completely.
18. If z is a complex number and $(z + 1)(2 - i) = 3 - 4i$ find z in standard form.
19. Solve for x and y:

    $$-4 + (x + y)i = 2x - 5y + 5i$$

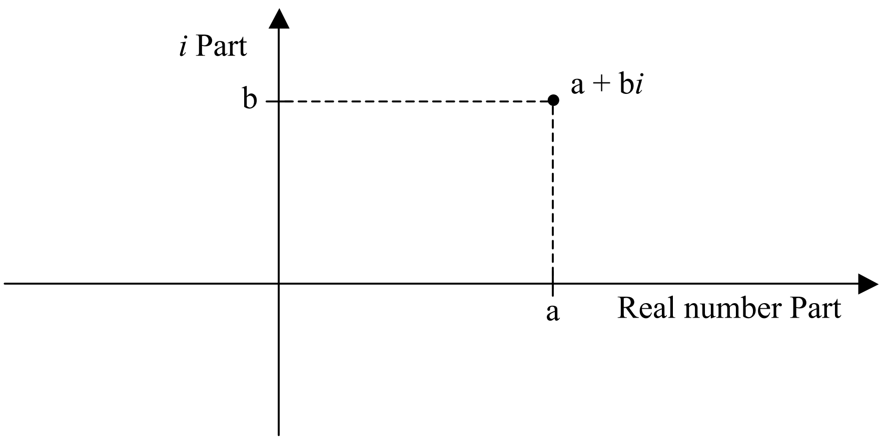
20. $1 + i$ is a root of $z^3 - 126z^2 + 250z - 248 = 0$ (Fact). Find the real root.
21. Show that $x - i$ is a factor of $x^3 + hx^2 + x + h$ regardless of the value of h.

Exercise 8.2 Answers

2. $x = \dfrac{-1+\sqrt{3}i}{2}$ *or* $\dfrac{-1-\sqrt{3}i}{2}$
3. a) 3i b) 4i c) -12 d) 1 e) 1 f) -1
4. 2
5. 16
6. $\frac{1}{2}-\frac{1}{2}\mathrm{i}$
7. $5 - i$
8. 18i
9. i
10. 2, -2i, 2i, -2
11. No.
12. $z = 1 + i$ or $1 - i$
13. 2, 3, 4
14. a = 2, b = -3 or a = -2, b = +3
15. a) i b) 2 c) 0
16. 24
17. $z = 2 + i$ or $2 - i$ or $-2 + i$ or $-2 - i$
18. $1 - i$
19. x = 3, y = 2
20. Other root is +124

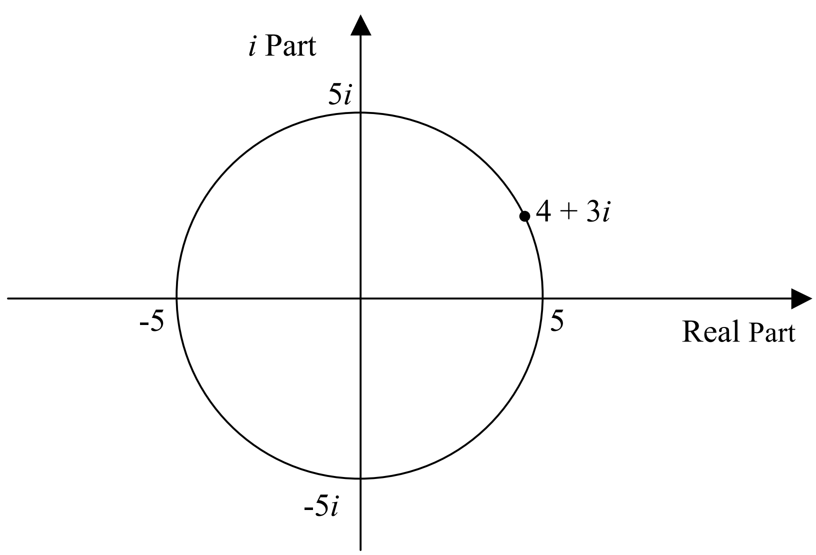
Argand Diagram

As has been proven, complex numbers cannot be represented on a number line and are often represented on an Argand Diagram as shown below.



|z| is the magnitude or modulus of z and means the distance of z from the origin on the Argand Diagram. It therefore follows that |z| is a real number since it represents a distance. For example $|2 + 3i| = \sqrt{13}$ which is similar to saying $|\overrightarrow{(2,3)}| = \sqrt{13}$

The set represented by the following $\{z : |z| = 5\}$ is therefore a circle since the set contains those complex numbers on the Argand Digram which are 5 units from the origin. For example $4 + 3i$ is in the set.



Exercise 8.3

Argand Diagram

1. Solve $z^4 = 1$ where z is a complex number. Graph the four roots on an Argand Diagram.
2. What is the distance between $2 + 3i$ and $5 + 7i$ on the Argand Diagram?
3. Solve $z^3 = 1$ where z is a complex number. Graph the three roots on a Argand Diagram.
4. Using questions 1 and 3 as hints try to guess the roots of $z^8 = 1$. Check by multiplication to see if your guesses are correct.
5. Find a) $|4 – 3i|$ b) $|4 + 3i|$ c) $|3 – 4i|$ d) $|3 + 4i|$ e) $|1+i|$ f) $|1 – i|$
6. Try to find a complex number z, other than 13 such that $|z| = 13$.
7. Graph on an Argand Diagram the set $\{z\in C, |z| = 5\}$
8. Graph on an Argand Diagram the set $\{z\in C, |z| = 4\}$
9. Graph on an Argand Diagram the set $\{z\in C, |z –1| = 4\}$
10. Graph on an Argand Diagram the set $\{z\in C, |z – i| = 4\}$
11. Graph on an Argand Diagram the set $\{z\in C, |z – 1| = z + 3\}$
12. If $|z| = \sqrt{13}$ and the real part of z is 2 write down z in the form of $a + bi$.
13. Convince yourself that if $a + bi = 3 + 4i$ then $a = 3$ and $b = 4$ is the only possible solution. Use this fact to express $\sqrt{-7+24i}$ in the form $a + bi$. Hint let $\sqrt{-7+24i}$ be $a + bi$ and square both sides of the equation.
14. $\{z : |z – i| = 2\}$ represents the same circle as $\{z : |z + 3i| = m|z|\}$ Find the values of m.

Exercise 8.3 Answers

1. 1, -1, i, $-i$ 2. 5 3. 1, $\frac{-1-\sqrt{3}i}{2}$, $\frac{-1+\sqrt{3}i}{2}$

4. 1, -1, i, $-i$, $\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i$, $\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i$, $-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i$, $-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i$

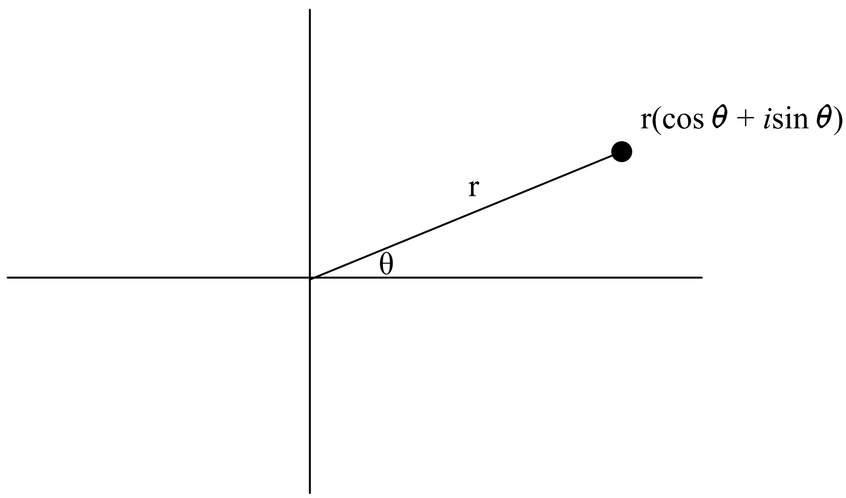
5. a) 5 b) 5 c) 5 d) 5 e) $\sqrt{2}$ f) $\sqrt{2}$ 6. $5 + 12i$

12. $2 + 3i$ 13. $3 + 4i$ 14. $m = 2$

Polar Form of a Complex Number

A complex number may be represented on the Argand Diagram not only in the standard form of a + bi but also as r(cos θ + $i$sin θ) where r represents the distance of the complex number from the origin and θ is the angle which the line from the origin to the complex number makes with the positive real axis. θ is sometimes called the argument.

Argand Diagram



r(cos θ + $i$sin θ) is often abbreviated to rcis θ (especially in North America but less so elsewhere).

Note that r is always positive and θ is the angle in standard position.

When two complex numbers are multiplied the result yields a complex number whose magnitude is the **PRODUCT** of the **MAGNITUDES** of the two original complex numbers and whose angle made with the positive real axis is the **ADDITION** of the two angles of the original complex numbers.

i.e r(cos θ+ $i$sin θ) times s(cos α + $i$sin α)

= rs(cos(θ + α) + $i$sin(θ + α))

Example $2(\cos 30° + i\sin 30°)$ times $3(\cos 60° + i\sin 60°)$

$= 6(\cos 30°\cos 60° + \cos 30° i\sin 60° + i\sin 30°\cos 60° + i^2\sin 30\sin 60°)$

$= 6(\cos 30°\cos 60° – \sin 30°\sin 60° + i(\sin 60°\cos 30° + \cos 60°\sin 30°))$

$= 6(\cos(30° + 60°) + i\sin(30° + 60°))$

$= 6(0 + i)$

$= 6i$

As an example $2(\cos 20° + i\sin 20°)$ times $5(\cos 70° + i\sin 70°)$

$= 10(\cos 90° + i\sin 90°)$

$= 10i$

It follow as a natural consequence that when dividing complex numbers we divide the magnitudes and subtract the angles.

<u>Example</u>

$$\frac{6(\cos 70° + i\sin 70°)}{2(\cos 30° + i\sin 30°)} = 3(\cos 40° + i\sin 40°)$$

<u>de Moivre's Theorem</u>

A most important theorem in complex numbers is de Moivre's Theorem which states that

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

Let's see a simple example.

$[2(\cos 30° + i\sin 30°)]^3 = 8(\cos 90° + i\sin 90°)$

$= 8i$

It is left as an exercise for the student to see that $(\sqrt{3} + i)^3 = 8i$ by multiplication.

In fact de Moivre's Theorem is true for <u>**ANY**</u> value of n (positive, negative, fraction or otherwise). e.g. $[4(\cos 60° + i\sin 60°)]^{\frac{1}{2}} = 2(\cos 30° + i\sin 30°)$

Example

Roots of Unity

To solve $z^6 = 1$

Let $z = \cos\theta + i\sin\theta = \text{cis}\,\theta$

Then $z^6 = \cos 6\theta + i\sin 6\theta = \text{cis}\,6\theta$

$\therefore \cos 6\theta + i\sin 6\theta = 1 = \text{cis}0$ or cis 360° or cis 720° ….

$\therefore 6\theta = 0°$ or 360° or 720° or 1080° or 1440° or 1800°.

$\therefore \theta = 0°, 60°, 120°, 180°, 240°$ or 300°

$\therefore z = \text{cis}0°$ or cis60° or cis120° or cis180° or cis240° or cis300°

$\therefore z = 1$ or $\frac{1}{2}+\frac{\sqrt{3}}{2}i$ or $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$ or -1, or $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$ or $\frac{1}{2}-\frac{\sqrt{3}}{2}i$

Exercise 8.4

1. Express $1 + i$ in polar form
2. Express $2(\cos 60° + i\sin 60°)$ in standard form.
3. Express the following in polar form.

   a) $\frac{\sqrt{3}}{2}+\frac{1}{2}i$ b) $2 + 2i$ c) $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$
4. Express the following in standard form.

   a) $4(\cos 30° + i\sin 30°)$ b) 6cis120° c)2cis90° d) cis270°
5. Express $\text{cis}30° \cdot \text{cis}60°$ in simple form.
6. Write $\text{cis}\theta \cdot \text{cis}\alpha$ in polar form.
7. Express $(1 + i)^2$ in polar form.
8. Express $(1 + i)^4$ in polar form.
9. Express $(1 + i)^{20}$ in polar form.
10. If z = 2cis70° what is a) $|z|$ b) $\bar{z}$ c) $|\bar{z}|$?
11. Evaluate $(\frac{\sqrt{3}}{2}+\frac{1}{2}i)(\frac{1}{2}+\frac{\sqrt{3}}{2}i)$ in your head.

Exercise 8.4 (cont'd)

12. Evaluate $(\frac{\sqrt{3}}{2}+\frac{1}{2}i)^{32}$ in your head

13. Write $2\text{cis}70°\cdot 3\text{cis}(-40°)$ in standard form.

14. Multiplying a complex number by i in the complex plane is equivalent to a rotation of $\theta°$. State the value of θ.

15. Using polar form methods, write $i^{\frac{2}{3}}$ in standard form.

16. Express $(1 - i)^7$ in standard form

17. Solve $z^4 = -8-8\sqrt{3}i$ expressing the roots in the form a + bi

18. Express $-2\sqrt{3}+2i$ in polar form

19. Solve $z^5 = 1$ using polar form methods i.e. de Moivre's Theorem. Write the roots in polar form.

20. Solve $z^3 = i$. Write the three roots in standard form.

21. Simplify a) $\frac{12\text{cis}75°}{3\text{cis}15°}$ b) $\frac{18\text{cis}30°}{3\text{cis(-30)}°}$

Exercise 8.4 Answers

1. $\sqrt{2}\text{cis}45°$ 2. $1+\sqrt{3}i$ 3. a) cis30° b) $2\sqrt{2}$ cis45° c) cis120°

4. a) $2\sqrt{3}+2i$ b) $-3+3\sqrt{3}i$ c) $2i$ d) $-i$ 5. i 6. $\text{cis}(\theta+\alpha)$

7. 2cis90° 8. 4cis180° 9. 1024cis180°

10. a) 2 b) 2cis(-70°) c) 2 11. i 12. $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$ 13. $3\sqrt{3}+3i$

14. 90° 15. $\frac{1}{2}+\frac{\sqrt{3}}{2}i$ 16. 8+8i 17. $1+\sqrt{3}i$

18. 4cis150° 19. 1 or cis72° or cis144° or cis216° or cis288°

20. $\frac{\sqrt{3}}{2}+\frac{1}{2}i$ or $-\frac{\sqrt{3}}{2}+\frac{1}{2}i$ or -i

21. a) 4cis60°

b) 6cis60°

Exercise 8.5

1. Graph on an Argand Diagram the set $\{z \in C, 1 \leq |z| \leq 2\}$
2. Graph on an Argand Diagram the set $\{z \in C, z + \bar{z} = 2\}$
3. Solve $z^3 - (3 + i)z + 2 + i = 0$
4. Solve for x and y

   $(3 - 2i)(x + yi) = 2(x - 2yi) + 2i - 1$
5. Find a complex number z such that $z = (\bar{z})^2$
6. Evaluate $(\frac{\sqrt{3}}{2} + \frac{1}{2}i)^{18}$
7. Simplify $\frac{8\text{cis}140°}{2\text{cis}50°}$
8. Solve for z.

   $z^4 = -4$
9. Evaluate $|i^{101} - 1|$
10. Solve completely $z^3 - 2z = 4$ where $z \in$ Complex Numbers.
11. Does a quintic polynomial equation with real co-efficients always have a **real** root?
12. Solve completely $z^4 - 3z^3 - 6z^2 + 18z + 20 = 0$ where $z \in$ Complex Numbers
13. a) Draw the set $\{z : |z - 1| + |z - 5| = 8\}$ on an Argand Diagram
14. b) Draw the set $\{z : |z - 1| = |z - i|\}$ on an Argand Diagram
15. Solve the following equation where z is a complex number.

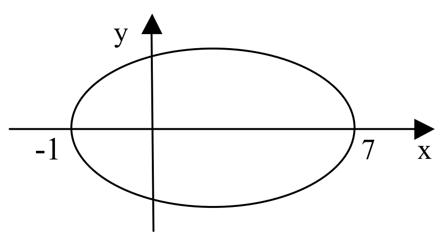
    $z^3 = 2 - 2i$
16. Using de Moivre's Theorem prove that

    $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$
17. Name the minimum positive integer n so that

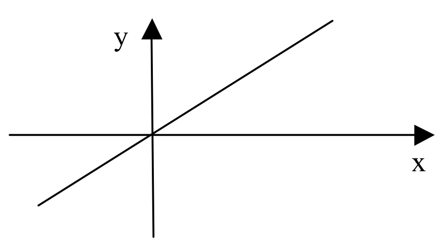
    $\left(\frac{3}{7} + \frac{3}{7}i\right)^n$ is a real number.

Exercise 8.5 Answers

3. 1, 2 + *i*
4. x = -1, y = 0
5. cis120°
6. -1
7. 4*i*
8. 1 + *i*, 1 – *i*, -1 + *i*, -1 – *i*
9. $\sqrt{2}$
10. 2, -1 + *i*, -1 – *i*
11. Yes
12. -1, -2, 3 + *i*, 3 – *i*
13. a)



b)



14. $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$
15. $\sqrt{2}\text{cis}105°, \sqrt{2}\text{cis}225°, \sqrt{2}\text{cis}340°$
17. n = 4